Add the rows. Add the columns. Add the diagonals.

A

8	1	6	
3	5	7	
4	9	2	

B

10	12	2	
0	8	16	
14	4	6	

C

2	11	5	
9	6	3	
7	1	10	

When the row, column, and diagonal sums are the same, we say the square is MAGIC.

Fill in the green blocks so that each square of 9 green blocks is MAGIC. Show the row, column, and diagonal sums in the blue blocks.

D

1		3	12
	4		
5		7	
12		12	

E

	5		
1	4		
6	3		

F

		3	6
1		2	

G

		1	
2			
		2	
	3		

H

			6
2			
1		1	

I

	10		
18		9	
	30		

Look for a connection between the number in the center block and the Magic total. In each row, column, and diagonal that includes the center block, what do you notice about the other two numbers in that row, column, or diagonal?

J

		21	
	27	6	
			45

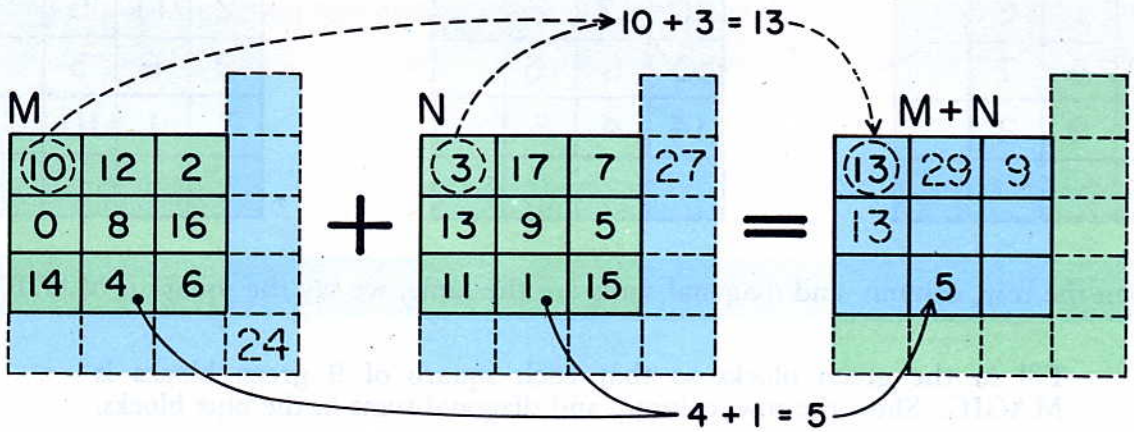
K

3	19		
	1		
			30

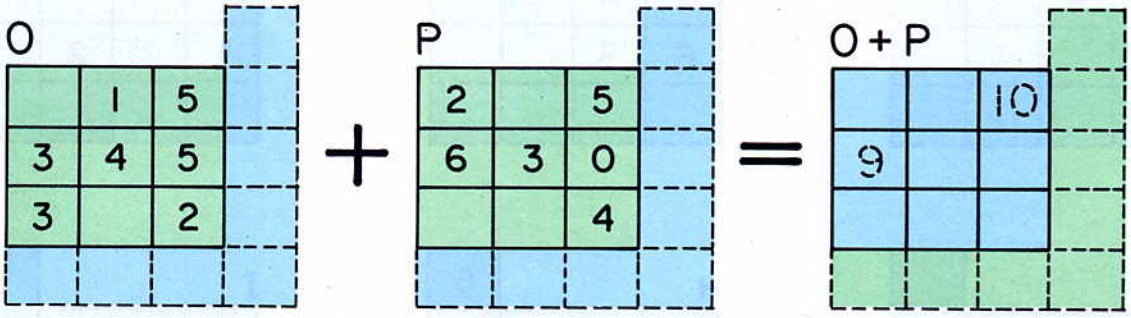
L

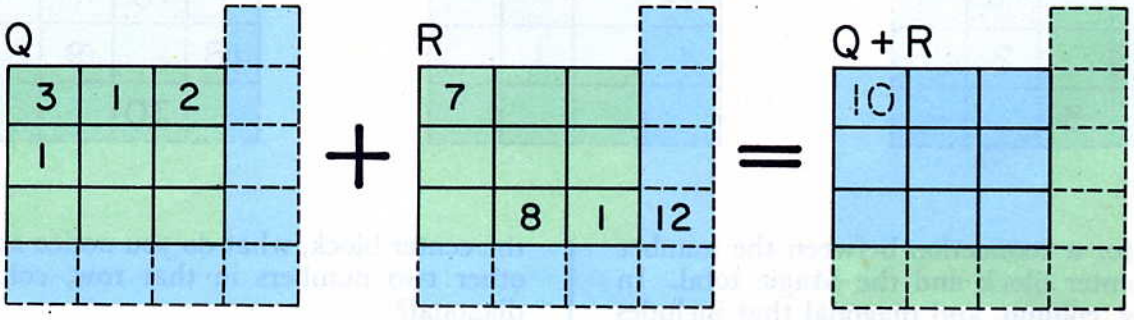
	0		
0		0	

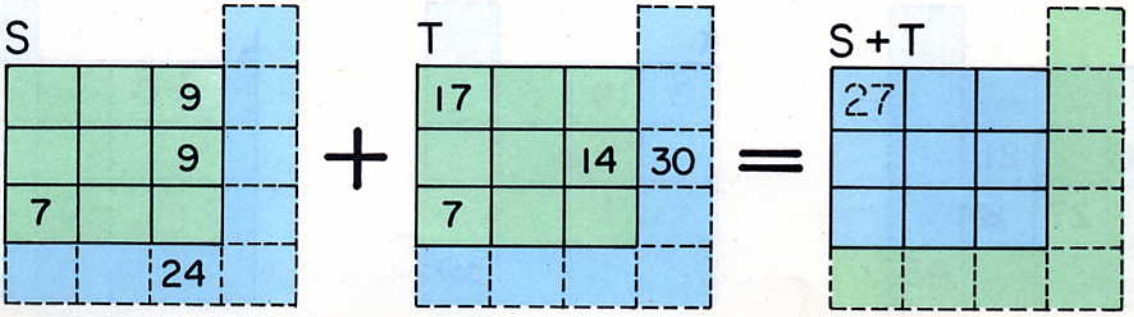
Harry told the class, "I've found an easy way to make a new Magic Square out of two old ones. I just add the old ones — that is, I just . . . well, here's what I mean!"



Let's try Harry's idea on other Magic Squares. Complete each of the first two (tinted green) as a Magic Square. Then "add" them as Harry did. Is the result also a Magic Square? You may want to try others on your own.

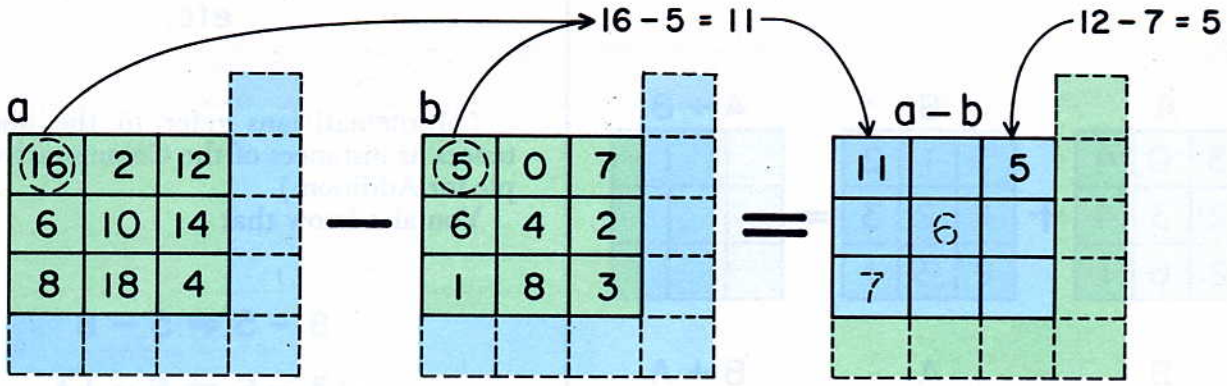


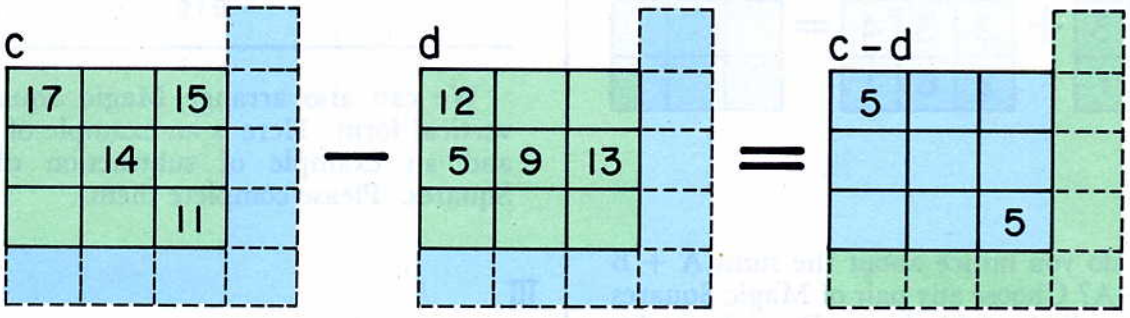


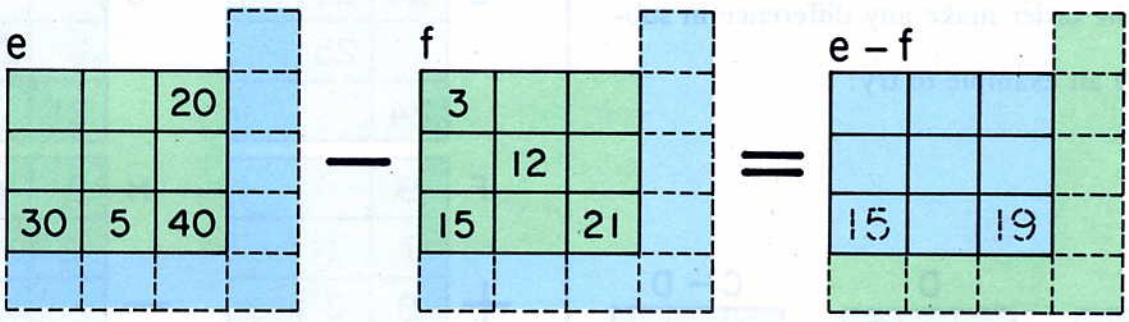


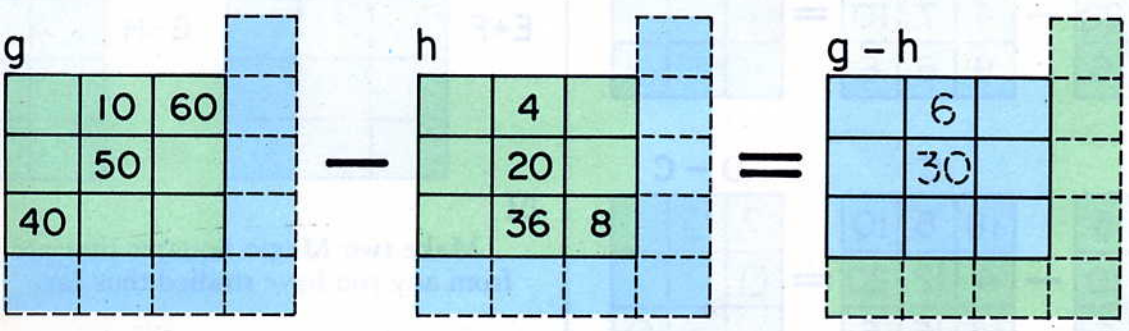
After Ellen had tried some examples, she exclaimed, "I can do the same thing with subtraction. It always works!"

Please finish the examples she gave.









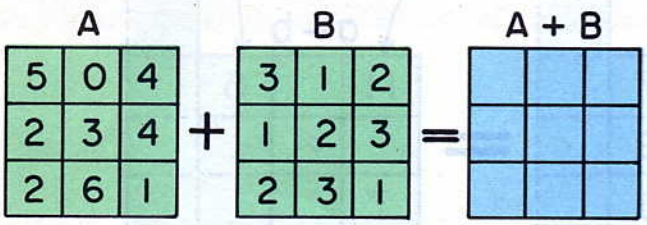
Helen asked, "Does it make any difference in which order you add a pair of Magic Squares? I know that $3 + 9 = 12$ and $9 + 3 = 12$, so I can say that $3 + 9 = 9 + 3$."

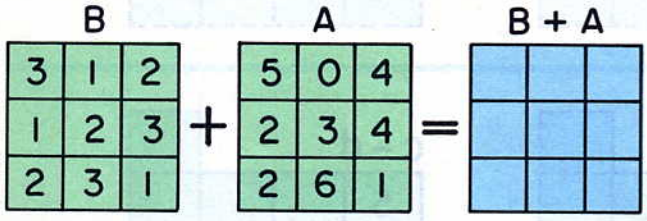
Would the same kind of rule be true of Magic Squares?

Let's try an example.

Below we can add a pair of Magic Squares in two different ways. Does the same Magic Square result?

I.



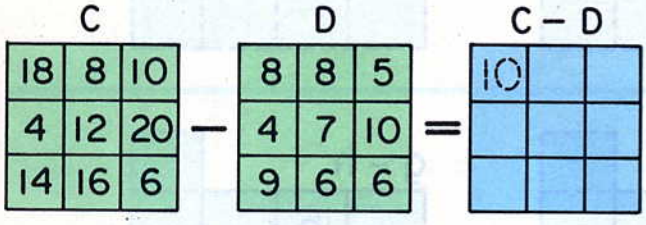


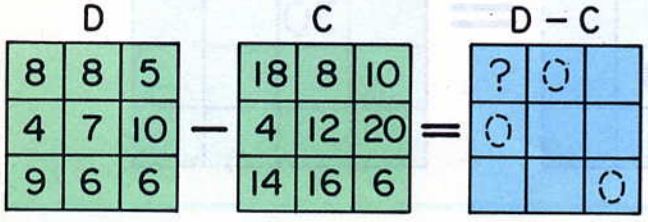
What do you notice about the sums A + B and B + A? Choose any pair of Magic Squares already studied. Add them. Does the order make any difference?

Does the order make any difference in subtraction?

Here is an example to try:

II.





Clearly, the rule does not hold for subtraction, but you expected that. You know that

$$3 + 7 = 7 + 3$$

$$9 + 5 = 5 + 9$$

etc.

(Mathematicians refer to the above sentences as instances of the Commutative Principle for Addition.)

You also know that

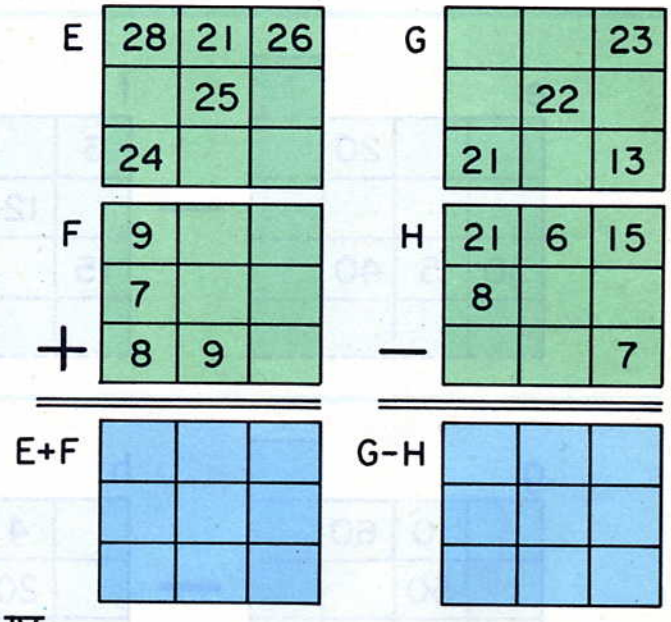
$$8 - 5 \neq 5 - 8$$

$$13 - 6 \neq 6 - 13$$

etc.

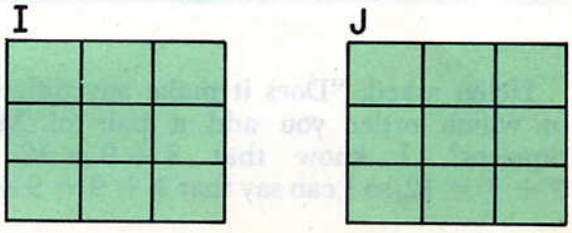
We can also arrange Magic Squares in a vertical form. Here is an example of addition and an example of subtraction of Magic Squares. Please complete them.

III.



IV.

Make two Magic Squares that are different from any you have studied thus far.



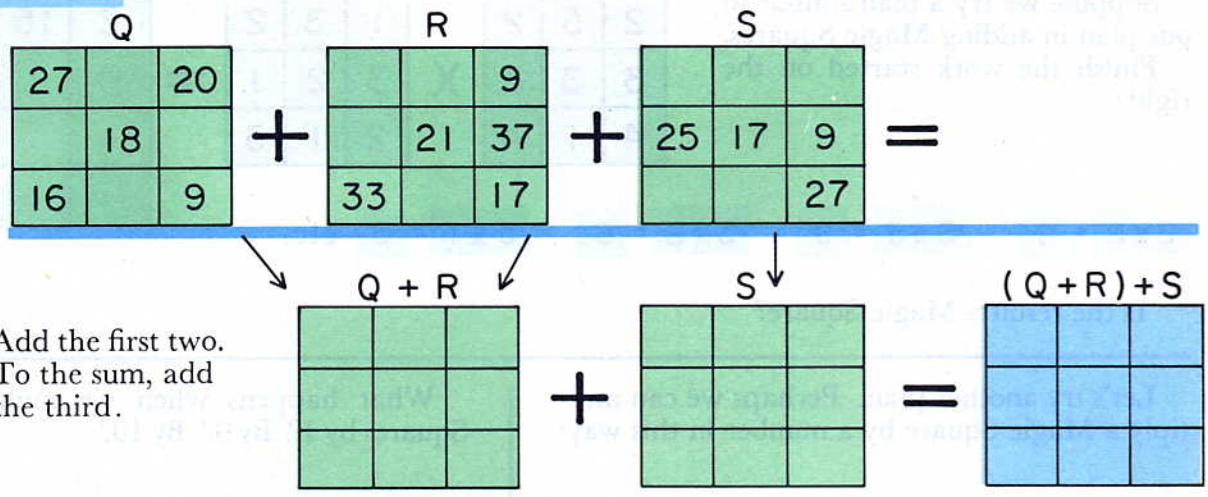
Warren asked whether we might not be able to add three Magic Squares. "Something like adding three numbers," he explained. "I wonder if the grouping would make any difference?"

What do you think?

Here is an experiment.

I.

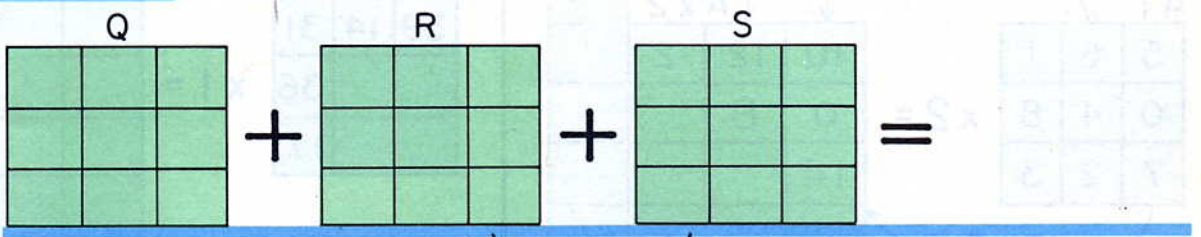
In one grouping



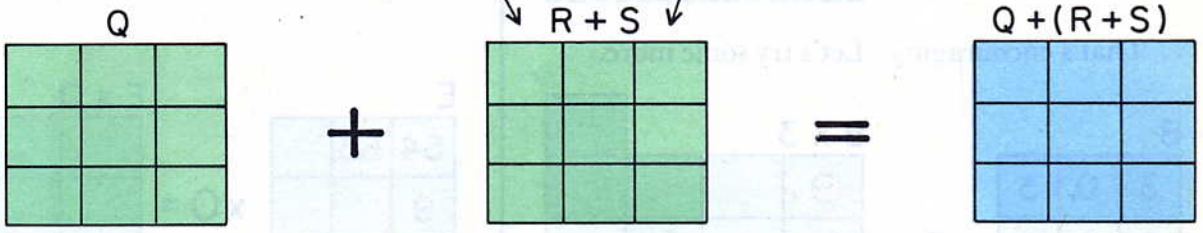
Add the first two. To the sum, add the third.

II.

In another grouping



Add the sum of the 2nd and 3rd to the first.



Is (Q + R) + S the same as Q + (R + S)?

III.

Try this same grouping experiment with other groups of 3 Magic Squares. Choose some we have already considered or make up some of your own.

Our results remind us of what mathematicians call the Associative Principle for Addition.

Here are two instances of this principle in adding numbers.

$$(8 + 7) + 3 = 8 + (7 + 3)$$
$$15 + 3 = 8 + 10$$

and

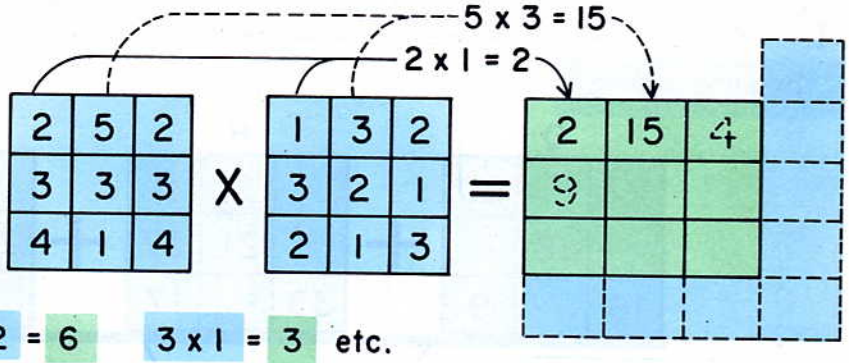
$$(10 + 7) + 9 = 10 + (7 + 9)$$
$$17 + 9 = 10 + 16$$

"How about multiplication?" Sarah asked. "We can always add Magic Squares and sometimes we can subtract one from another. Can we multiply one Magic Square by another?"

What do you think?

Suppose we try a plan similar to our plan in adding Magic Squares.

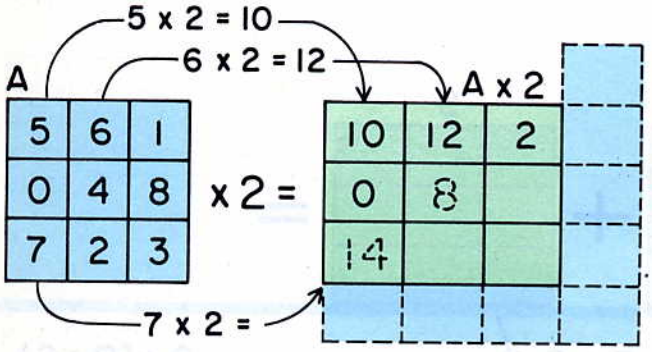
Finish the work started on the right:



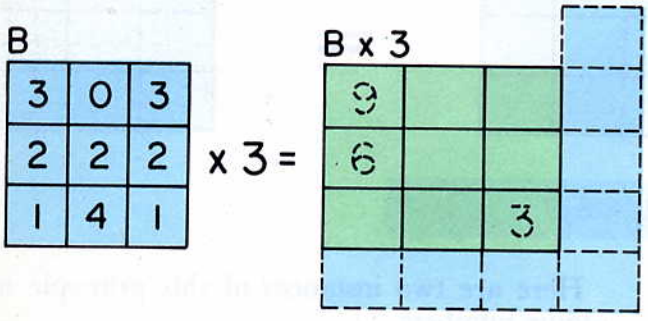
2 x 2 = 4 3 x 3 = 9 3 x 2 = 6 3 x 1 = 3 etc.

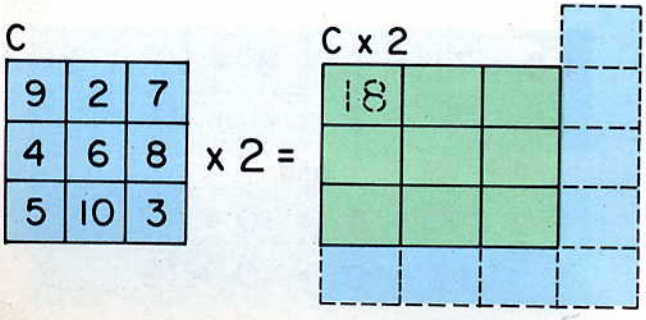
Is the result a Magic Square?

Let's try another plan. Perhaps we can multiply a Magic Square by a number in this way:

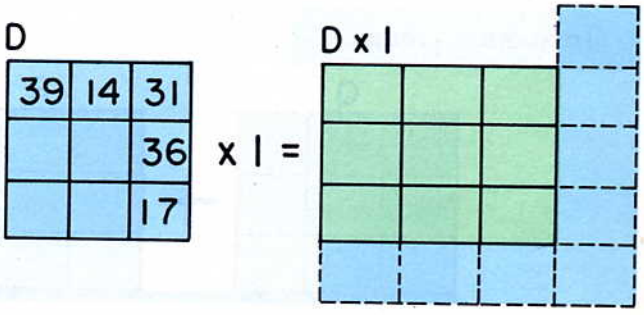


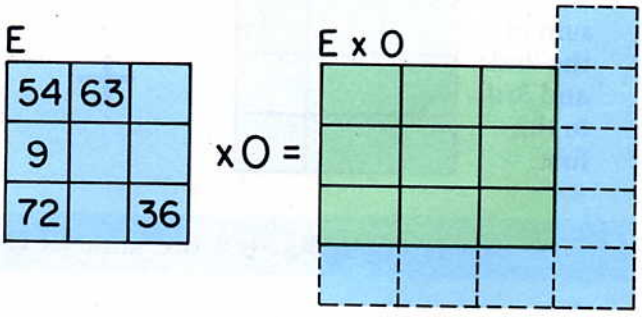
That's encouraging. Let's try some more.

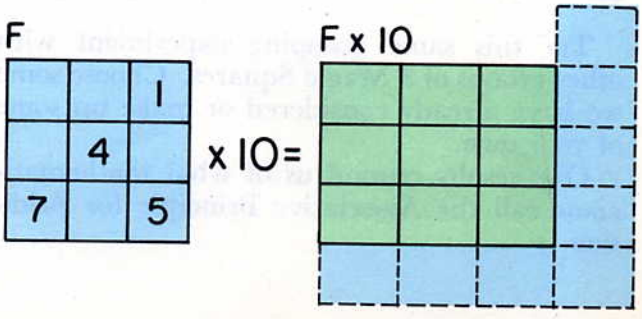




What happens when we multiply Magic Squares by 1? By 0? By 10?



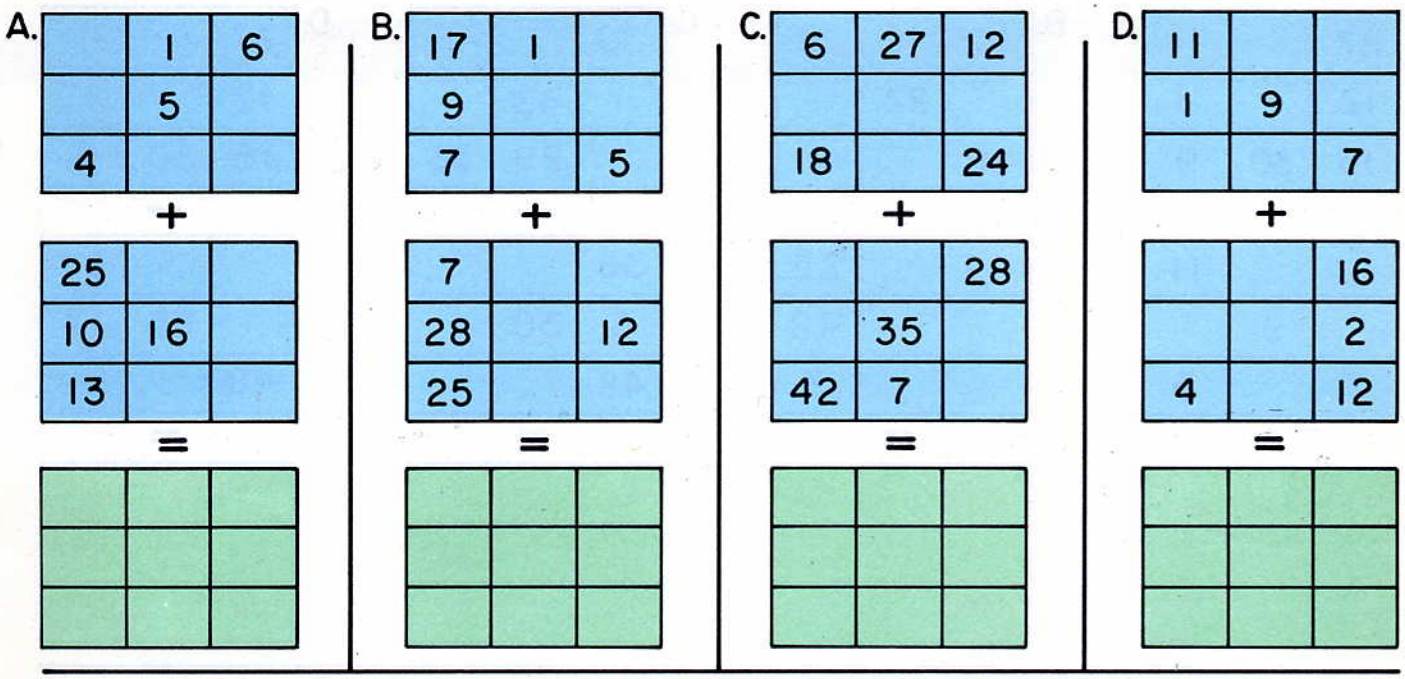




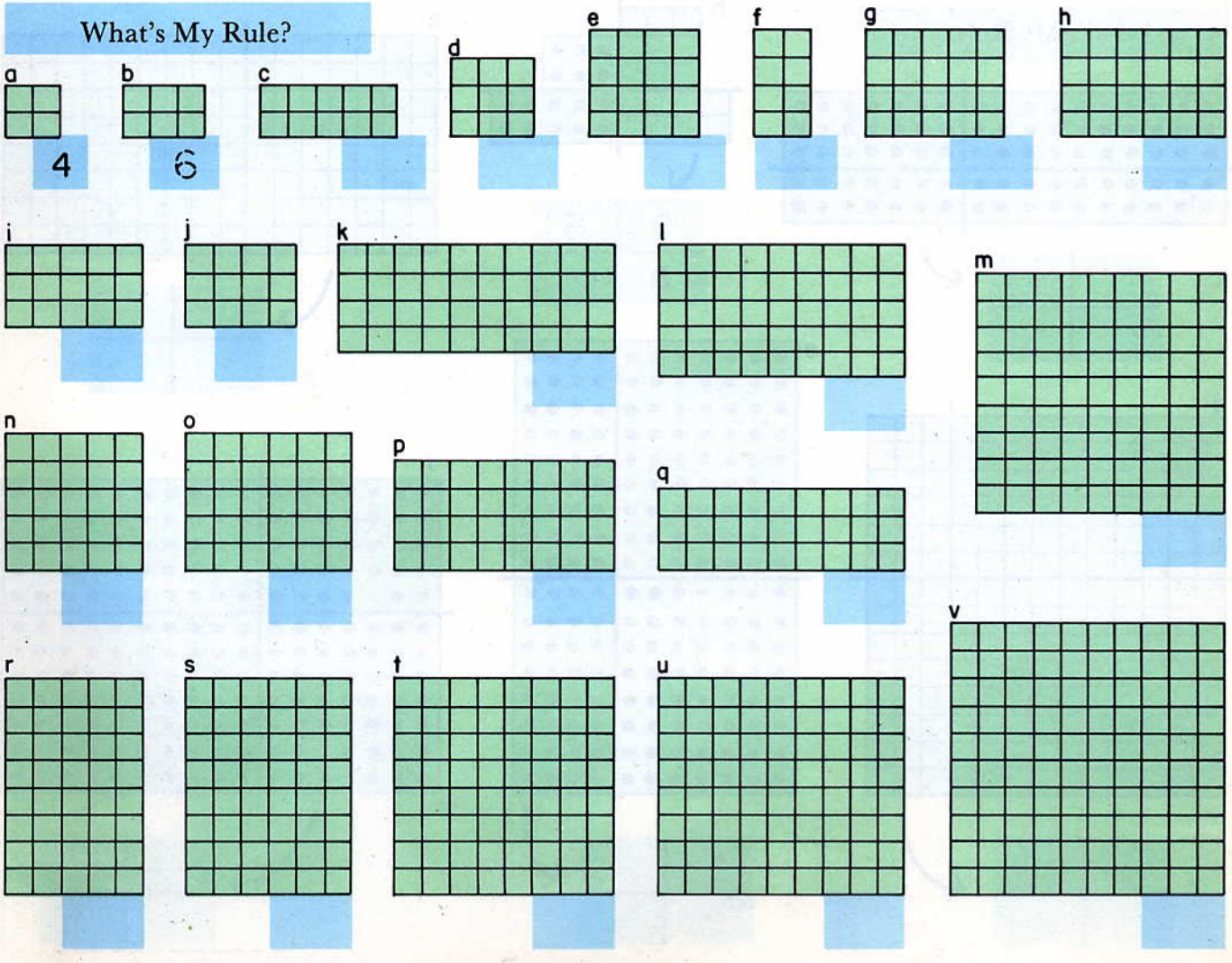
Finish each pair of blue-tinted boxes (A-D) to make Magic Squares.

Then add the squares in each pair to make a new one.

Summary: The sum of Magic Squares is (always, sometimes, never) a Magic Square.



What's My Rule?



Finish each pair of green-tinted boxes to make Magic Squares.

Then subtract the second square from the top square to make a new one.

Summary: The difference of a pair of Magic Squares is (always, sometimes, never) a Magic Square.

A.

27		
12		
15	30	9

−

		11
	9	
7		3

=

B.

40		26
	22	
18		4

−

49		35
		42
		7

=

C.

50		
	43	
	29	36

−

36		
6	30	
48		

=

D.

27		
12		
15	30	9

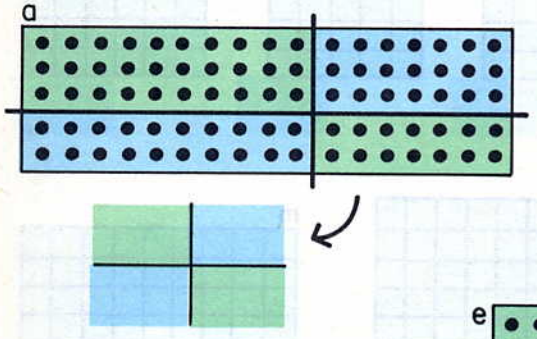
−

		62
	55	17
48		

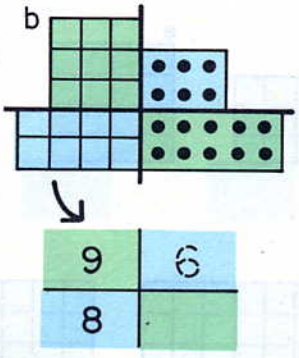
=

What's My Rule?

a

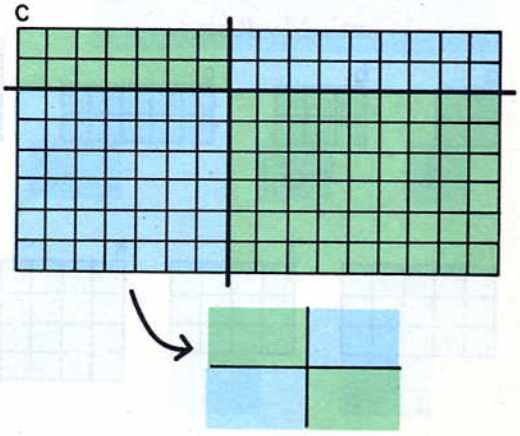


b

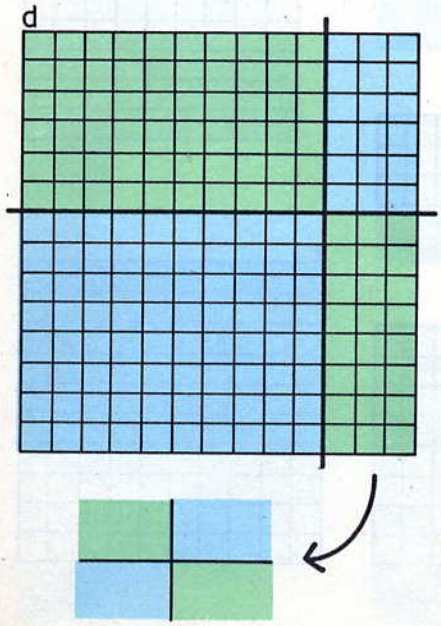


9	6
8	

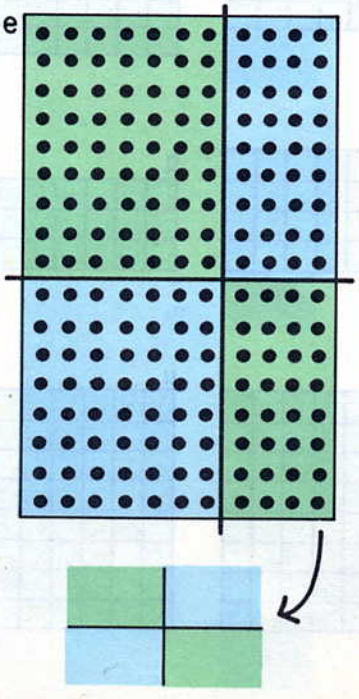
c



d



e



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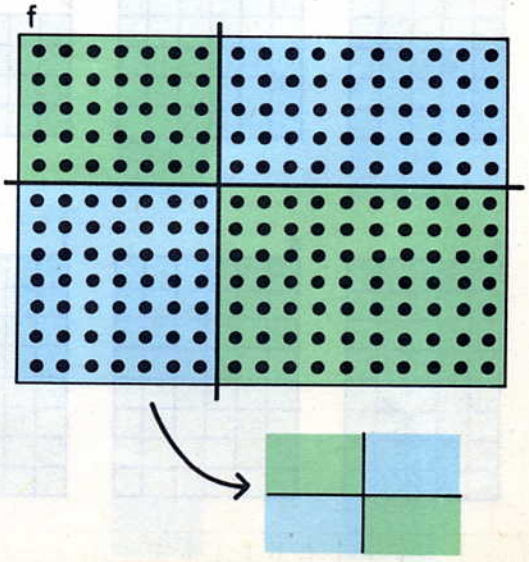


Summary:

The product of a Magic Square by a number is (always, sometimes, never) a Magic Square.

1.

8	9	4
3	7	11
10	5	6

x 3 =

		12
30		

2.

7	2	9
8	6	4
3	10	5

x 4 =

	8	

3.

1	6	5
8	4	0
3	2	7

x 6 =

6		

4.

10	45	20
35	25	15
30	5	40

x 2 =

20		

5.

16	2	12
6	10	14
8	18	4

x 5 =

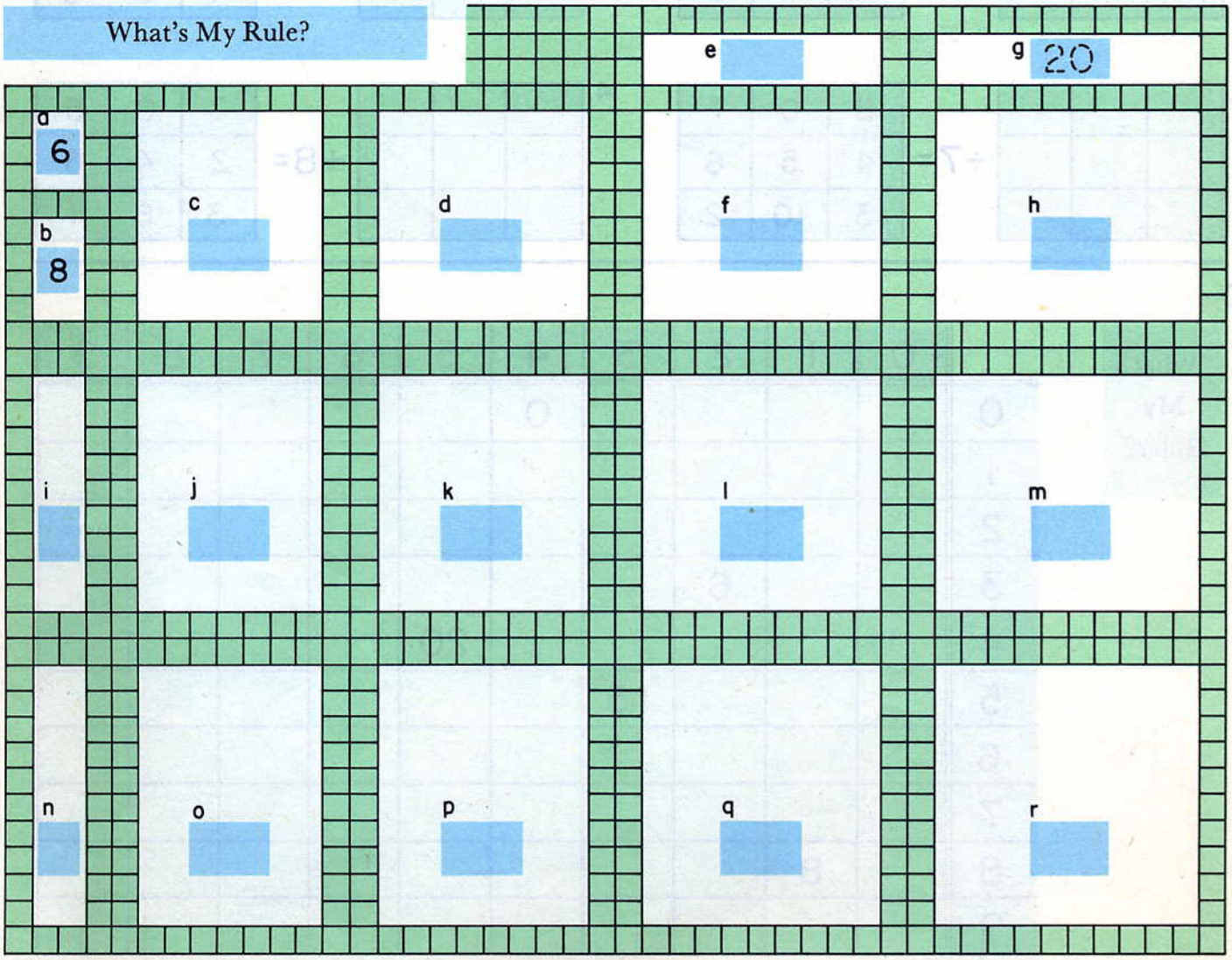
6.

3	8	1
2	4	6
7	0	5

x 7 =

14		

What's My Rule?



A Summary:

The quotient of a Magic Square by a number is (always, sometimes, never) a Magic Square.

1.

6	20	10
16	12	8
14	4	18

÷2=

3.

27	6	21
12	18	24
15	30	9

÷3=

2.

24	28	8
4	20	36
32	12	16

÷4=

4.

15	1	11
5	9	13
7	17	3

÷10=

B Can you rebuild Magic Squares in the green boxes?

1.

15		
25		

÷5=

3	8	7
10	6	2
5	4	9

3.

	12	

÷6=

7	2	6
4	5	6
4	8	3

2.

÷7=

8	0	7
4	5	6
3	10	2

4.

÷8=

7	0	5
2	4	6
3	8	1

C

"What's My Rule?"

	0	1	2	3	4	5	6	7	8	9
0					0					
1										
2										
3			6							
4						20				
5				15						
6										
7										
8		8								
9										

Following Directions

A.

	3	15	28	44	19
+ 5	8				

B.

	7	11	20	28	52
− 3	4				

C.

	2	5	9	0	10
x 3	6				

D.

	8	10	18	20	50
÷ 2	4				

E.

	12	21	47	65	99
− 8					

F.

	6	23	17	85	49
+ 9					

G.

	3	5	7	9	6
x 7					

H.

	8	12	19	49	88
+12					

I.

	2	3	5	8	0
x 9					

J.

	24	6	30	60	48
÷ 6					

K.

	16	28	20
+ 3	19		
x 2			40
÷ 4		7	

L.

	24	30	18
− 6			
+ 6		36	
÷ 3			6

M.

	8	10	15
x 4			
− 8			
+12			

N.

	11	6	9
x 3			
x 6			
x 9			

O.

	25	40	50
÷ 5			
+15			
− 6			

P.

	6	10	8
x 5			
+27			
x 8			

Q.

	21	49	56
−10			
÷ 7			
+16			

R.

	72	48	56
÷ 8			
− 9			
x 0			

S.

	45	27	54
x 2			
÷ 9			
−15			

A.

	12	24	6	30	18	36
÷6						
−5						
+9						
÷3						
x2						

B.

	20	32	16	40	36
−3					
÷2					
x2					
÷4					
−16					

C.

+	2	6	4	1	8	3	0	5	10	2	7	9	16
3		9											
1													
4										6			
0													
5			9			8							
7													
9													
6													
2				3									
8													

D.

X	2	6	4	1	8	3	0	5	10	2	7	9	16
3						9							
1										2			
4	8												
0													
5													
7													
9													
6													
2						6							
8													

Ben and Ruth Harris are Newcomers

Mrs. Brown introduced two new students to her class at Roosevelt School.

"I would like you all to meet Ben Harris and his sister, Ruth Harris. They moved to town this summer.

"Ben tells me that they have lived in a part of the mountains where there are no schools. Their mother has taught them at home.

"Did your mother have a class in arithmetic for you?" Mrs. Brown asked Ben.

"Well, she had a longer name for it; she called it a class in the Arithmetic of Things and Numbers.

"We learned how to add and subtract things, and we learned how to multiply and divide things by numbers."

The students in the class were truly puzzled by what Ben said. It sounded familiar and yet somewhat strange at the same time. They were glad that Mrs. Brown asked more questions.

"Did you learn to add 2 and 6?"

"Two what and six what?" Ben asked. "You see, we only learned how to add and subtract things!"

"Can you explain what you mean?"

Ben Explains

"You see, we learned that

2 stars + 6 stars = 8 stars
2 feet + 6 inches = $2\frac{1}{2}$ feet or 30 in.
2 cows + 6 pigs = 8 animals

and we learned that there isn't a very sensible answer to the question:

2 miles + 6 quarts = ?

"So, you see, I need to know what things you are talking about."

"Tell us about subtraction," Mrs. Brown suggested. "How much is seven minus two — or, seven take away two?"

Ben scratched his head. "Again, I need to know your answers to 7 what? and 2 what?

"Mother taught us that

7 marbles − 3 marbles = 4 marbles
7 hours − 3 minutes = 6 hrs., 57 min.
7 yards − 3 feet = 6 yards

and she also taught us that there was no very sensible answer to:

7 cows − 3 chickens = ?

"Do you see why I need to know what things it is you are talking about?"

Mrs. Brown smiled. "What you say certainly makes sense."

As we continue telling you the story of the newcomers — Ben and Ruth — we will leave blanks. Please fill in the blanks so that the story will make sense to anyone who might be reading it.

You will need to learn the special ways that Mrs. Harris taught Ben and Ruth.

Mrs. Brown turned to Ruth. "How much is 4 multiplied by 3?"

"Ben and I would have to ask you — 4 what? Here are the kinds of things we learned:

4 inches x 3 = 1 foot
4 ¢ x 3 = 12 ¢
4 quarters x 3 = $3.00
4 days x 3 = 1 wk. and ___ days
4 eggs x 3 = ___ dozen eggs

"We learned there's no sensible answer to:

4 stars x 3 stars = ?

"In division we learned this kind of idea:

1 doz. eggs ÷ 3 = 4 eggs
1 gallon ÷ 4 = ____ quart
1 week and 3 days ÷ 2 = ____ days
12 chocolates ÷ 3 = ____ chocolates
20¢ ÷ 4 = ____ ¢
1 hour ÷ 3 = ____ minutes

"Mother used to put the general idea in this way:

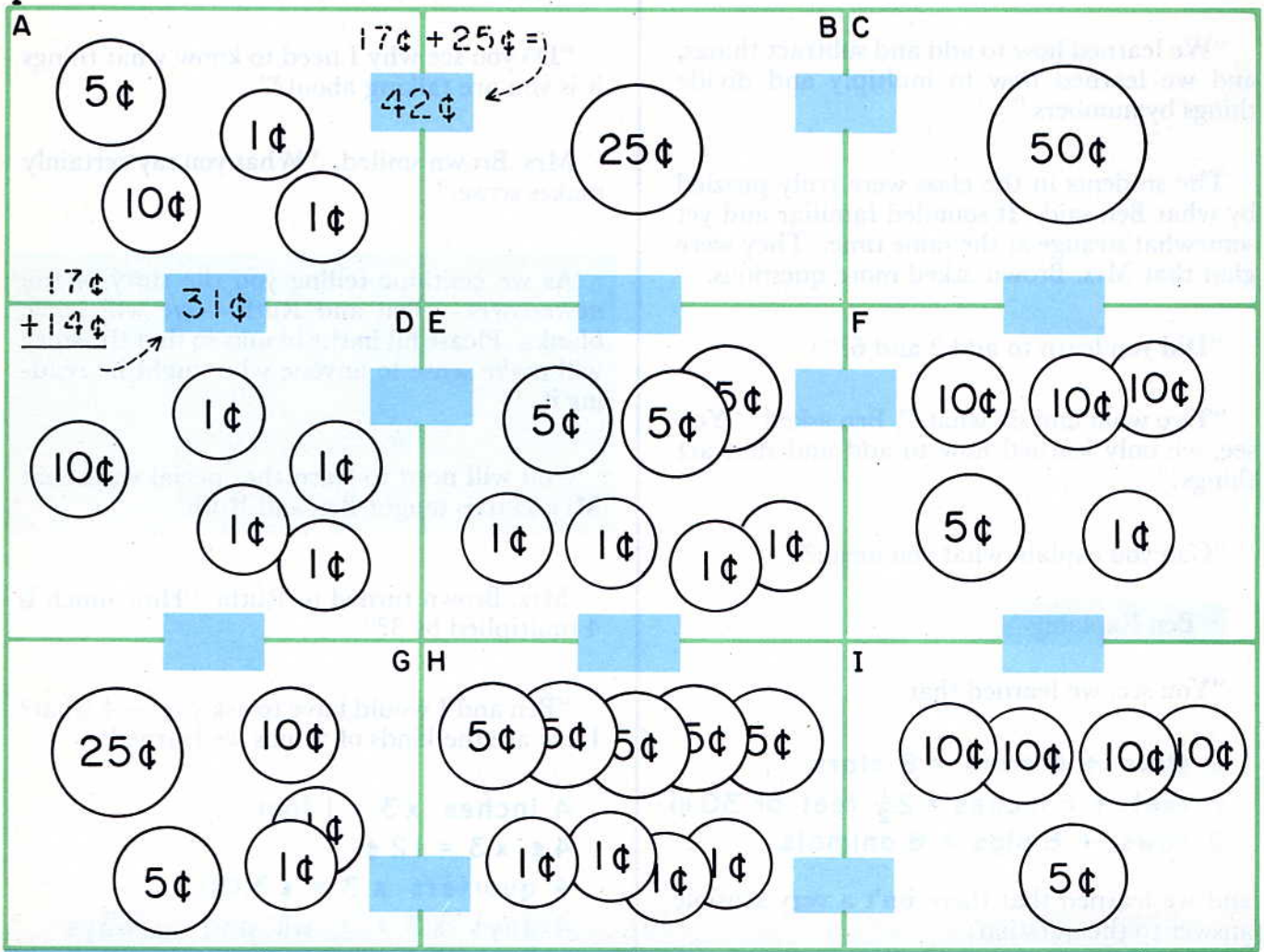
Things + Things = Things
Things − Things = Things
Things x a number = Things
Things ÷ a number = Things

"And she said it is most important to think about what you are saying so you can be sure you are talking sense — and not nonsense."

Mrs. Brown visited Mrs. Harris and talked about her "Arithmetic of Things and Numbers." Mrs. Brown asked to see some of the work that Ben and Ruth had done.

The pages that follow are samples Mrs. Harris gave to Mrs. Brown. Please fill in all the blanks you can.

I



II

a	b	c	d	e	f
29¢	78¢	61¢	23 miles	46 pounds	54 minutes
+32¢	−32¢	+19¢	− 8 miles	+46 pounds	−17 minutes
¢			miles		

Please fill in the blanks so that the sentences make sense.

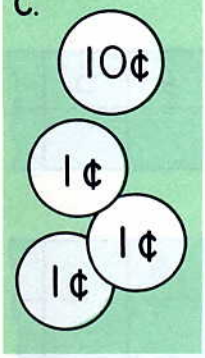
A.

4 nickels + 3 dimes = 50 ¢
1 hour − 10 minutes = ____ minutes
1 week + 6 days = ____ days
2 feet + 8 inches = ____ inches
1 quarter + 2 quarters = ____ ¢
1 yard − 12 inches = ____ feet
38 minutes + 17 minutes = ____ min.
17 pennies + 9 nickels = ____ ¢

B.

13 days + 8 days = ____ weeks
1 gallon − 1 quart = ____ quarts
2 weeks − 9 days = ____ days
$1.50 − 75¢ = ____ ¢
7 nickels + 9 nickels = ____ dimes
80 minutes + 40 minutes = ____ hrs.
2 pounds − 20 ounces = ____ ounces
8 yards − 19 feet = ____ feet

C.



. . . + a quarter = 38 ¢
. . . + 47¢ = ____ ¢
. . . − 4¢ = ____ ¢
. . . + 7 nickels = ____ ¢
. . . + 3 quarters = ____ ¢
. . . + 12 dimes = $ ____

. . . x 2 = 26 ¢
. . . x 3 = ____ ¢
. . . x 5 = ____ ¢
. . . x 8 = $ ____
. . . x 4 = ____ ¢
. . . x 6 = ____ ¢

Please fill in the blanks so that the sentences make sense.

D.

3 days x 2 = ____ days
13 minutes x 2 = ____ minutes
13 minutes x 3 = ____ minutes
14 eggs ÷ 2 = ____ eggs
1 foot ÷ 2 = ____ inches
1 nickel x 4 = ____ ¢
6 inches x 4 = ____ feet
30 minutes x 4 = ____ hours

E.

1 hour ÷ 2 = ____ minutes
1 hour ÷ 3 = ____ minutes
1 hour ÷ 4 = ____ minutes
1 gallon ÷ 2 = ____ quarts
2 feet ÷ 3 = ____ inches
28¢ ÷ 2 = ____ ¢
12 chocolates ÷ 3 = ____ chocolates
18 miles ÷ 3 = ____ miles

F.

1 week	$1.17	2 feet	90 minutes	5 yards
−3 days	+ 1.35	− 18 inches	−35 minutes	+7 feet
____ days	$ ____			____ feet

G.

15 miles	¢	6 inches	ounces	30 min.
x 2	3) 75¢	x 4	4) 2 pounds	x 6
____ miles		____ feet		____ hrs.

Ruth explained how she and Ben used to make charts for their mother.

"Sometimes Mother would write very easy questions for us to answer. Here is a sample:

How many quarts in 3 gallons ?

"Or she might ask us to complete a sentence like the following:

3 gal. = ____ quarts

"That's so easy that Ben and I made a list of answers to many sentences of the form:

▲ gal. = ■ qts.

"We might start the list this way:

Plan A.

gal.	0	1	2	3	4	5	6	etc.
qts.	0	4	8					

"You can fill in the blanks. We would put an arrow to show Mother where she could look for the numbers for the easy sentence she had asked us to complete.

"Sometimes we used a different plan for the top line in our list. As you fill in the blanks, you may see why we chose this plan.

Plan B.

g	1	2	3	5	8	13	etc.
q	4	8					

"Or, we might decide on a different plan. I'll start it. Please finish the top line and fill in the blanks."

Plan C.

g	1	2	4	8				
q	4	8						

"Here's another plan."

Plan D.

g	$\frac{1}{2}$	1	$1\frac{1}{2}$	2				
q	2							

"Show us another example," Bill suggested.

"All right," Ruth said. "I think you would agree that this sentence:

2 nickels = ____ pennies

is much too easy to worry about. So Ben and I would make up lists like these. I am sure you can complete them . . . and, if you like, extend them."

▲ nickels = ■ pennies

Plan E.

nickels	0	1	2	3	4	5	
pennies							

Plan F.

n	2	4	6	10	16		
p							

Plan G.

n	1	2	4	8			
p							

"Here's another question that's too easy:

How many days in 3 weeks ?

"So, we made several lists for Mother. We thought of a different kind of sentence:

▲ weeks = ■ days

Plan H.

weeks	0	1	2	3	4	5	
days	0	7					

Plan I.

w	3	6	12	24	48		
d							

"You may want to extend these lists or make some with plans of your own."

The children in Mrs. Brown's class wanted more examples; so Ruth made up several:

Mother's question:

If cakes are packed a half-dozen in a box, how many cakes are there in 3 boxes?

Our sentence:

▲ half-dozens = ▭ cakes

Plan J.

half-doz.	0	1	2	3	4	5
cakes						

Plan K.

half-doz.	5	10	15	20	25	30
cakes						

Plan L.

half-doz.	4	2		8	14	
cakes			36			42

Jumping around (with some planning).

half-doz.	6		16		8	28
cakes		60		120		

Mother's sentence: 1 foot = ____ inches

Our sentence: ▲ feet = ▭ in.

Plan M.

feet	0	1	2	3	4	5
inches						

Plan N.

ft.	2	4	6	10	16	26
in.						

Jumping around (with some planning).

ft.			12	3	20	23
in.	24	120				

Mother's sentences:

1 gallon = ____ half-gallon
1 gallon = ____ quarts
1 gallon = ____ pints
1 gallon = ____ half-pints

Our list shows the completions for these sentences — and many more sentences.

Plan O.

gallons	1	2	3	4	5	6
half-gal.	2					
quarts	4					
pints	8					
half-pints						

Plan P.

gal.	4	3	7	10	17	27
$\frac{1}{2}$ gal.						
qts.						
pts.						
$\frac{1}{2}$ pts.						

Mother's sentence: 2 hours = ____ minutes

Too easy! So, we make tables like the following:

Plan Q.

hours	1	2	3	4	5
minutes	60				

Plan R.

hr.	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$
min.				60		

Plan S.

hr.	$\frac{1}{3}$	$\frac{2}{3}$	1	$1\frac{1}{3}$	$1\frac{2}{3}$	2	$2\frac{1}{3}$
min.			60				

"Of course," Ruth explained, "Mother sometimes gave us problems that were more interesting. Here is one:

1 nickel + 3 cents = ____ cents

"The answer is easy enough, but we wanted to give more than just an easy answer. So, we made a chart in the form of:

____ nickels + ____ cents = ... cents

Plan A.

+	0¢	1¢	2¢	3¢	4¢	5¢
0 nickels	0¢	1¢				
1 nickel	5¢			8¢		
2 n	10¢		12¢			
3 n						

"Please fill in the rest of the chart."

"You notice," Ruth said, "that our chart includes the answer to Mother's problem — in the heavy black box.

"Here is another plan we might use for a chart."

Plan B.

+	2¢	4¢	6¢	8¢	10¢	12¢
2n	12¢	14¢				
4n					30¢	
6n	32¢					
8n						

Plan C.

+	3¢	7¢	11¢	15¢	19¢	23¢
1n						
3n						
5n						
7n						

"Here is another one of mother's problems:

2 gals. + 3 qts. = ____ qts.

"We would make a chart in the form of:

____ gals. + ____ qts. = ... qts.

"To save space, we would use **g** for gallons and **q** for quarts.

Plan D.

+	0q	1q	2q	3q	4q
0g	0q	1q			
1g	4q				
2g					
3g					
4g					

"The answer to Mother's problem is in the heavy black-line box."

Jumping around

+	0q	4q	7q	10q	13q
5g	20q				
10g					
6g				34q	
12g					
7g					
14g					
8g					

"You might like to extend some of the charts — in both directions — down and to the right.

"Can you imagine a chart we might have made for this problem?"

3 dimes + 2 nickels = ____ cents

Frank wanted to know about the charts Ruth and Ben might have made when their Mother gave them problems involving subtraction.

"I will show you examples. But I want to warn you — we were not always able to fill in all the boxes in some of our charts. We drew large 'X's' in such boxes.

"Mother might say

3 dimes − 6 cents = ____ cents.

"Then Ben and I would make a chart of many ways of completing the sentence:

____ dimes − ____ cents = ... ¢

"We would outline the answer to Mother's problem with a heavy line."

Plan E.

—	0¢	3¢	6¢	9¢	12¢
1 dime	10¢	7¢		1¢	X
2 dimes		17¢			
3 d		27¢			
4 d				31¢	
5 d					

Jumping around.

—	7¢	14¢	25¢	31¢	18¢	42¢
2d	13¢		X	X		
3d				X		
5d						
8d						
6d						
9d						
10d			75¢			
11d						

"Mother might say

2 hours − 15 minutes = ____ min

and we included the answer as one of the entries in our chart. We can use 'm' as a shorthand for 'minutes.'"

Plan F.

—	15m	30m	45m	60m
1 h				
2 h				
3 h				
4 h				

Jumping around.

—	20 m	35 m	53 m	28 m
1 h				
1½ h				
½ h		X		
2½ h				

"Mother might say

3 weeks − 8 days = ____ days.

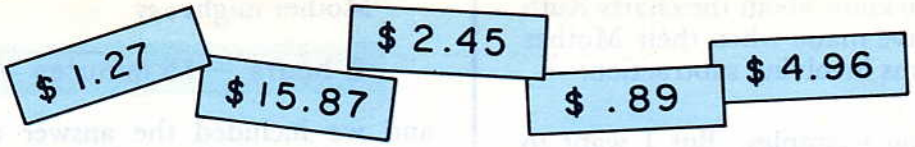
"We might give her the following answers."

Plan G.

—	2d	5d	8d	11d	14d
0 w					
1 w					
2 w					
3 w					

Jumping around.

—	7d	12d	18d	9d	23d
2 w		2d	X		
4 w					
6 w					
10 w					



Ben told the class about a puzzle his mother had made up.

"She would give us the price of several items — such as the five given above. If you bought 2 different items, how many different amounts could you spend?

"In this case, I think there are ten different ways. Can you find them all and name the amounts?"

A.

1. $ 1.27 + .89 = $ 2.16

2. $ 15.87 + 2.45 = $ 2

3. $ .89 + 4.96 = $

4. $

5. $

6. $

7. $

8. $

9. $

10. $

"Then Mother might ask if we could find as many different amounts buying 3 items at a time."

B.

11. $ 1.27 + 15.87 + 2.45 = $ 9

12. $

13. $

14. $

15. $

16. $

17. $

18. $

19. $

20. $

C.

How does the problem change if you buy 1 item at a time? . . . 4 items at a time? . . . Let's explore these problems.

C.

one at a time . . .

1.	2.	3.	4.	5.
$ 1.27	$	$	$	$

D.

four at a time . . .

1.	2.	3.	4.	5.
$	$	$	$	$
$	$	$	$	$

E. "Here is another example.

"Can you find 15 combinations of the prices of the 6 items shown — taking them 4 different prices at a time?

"How many totals are different?"

$5.12 $9.09 $3.37 $.91 $6.63 $4.88

1.	2.	3.	4.	5.
$ 5.12	$	$	$	$
.91				
3.37				
4.88				
$ 28	$	$	$	$

6.	7.	8.	9.	10.
$	$	$	$	$
$	$	$	$	$

11.	12.	13.	14.	15.
$	$	$	$	$
$	$	$	$	$

"Here is another kind of puzzle Mother invented for us," Ben exclaimed. "It's a kind of chain reaction. We might begin with 4 periods of time. We are to find the difference between each pair of given time periods.

"We start with:

316 minutes | 243 min. | 207 min. | 86 min.

"We remember we are talking about minutes, and we find the difference between each pair. Can you find all 6 pairs?

A.

1.	2.	3.	4.	5.	6.
316	243	316	243	316	
243	86	207			
73					

"Now, we have 6 new periods of time — given in numbers of minutes. Next, we consider all possible pairs of these new time periods — and find the difference in each case. Can you find 15 pairs?

B.

1.	2.	3.	4.	5.
157	230			
73	36			
84				

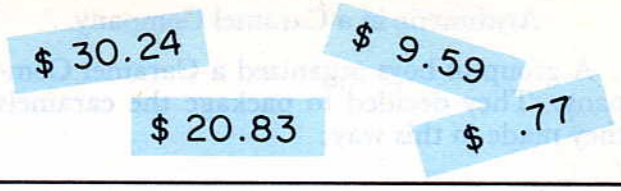
6.	7.	8.	9.	10.

11.	12.	13.	14.	15.

"With a twinkle in her eyes, Mother would ask if we wanted to keep going . . . that is, she asked if we wanted to consider each pair of the 11 new periods of time.

"We were anxious to go on. Perhaps you are, also. If you do go on, let me warn you we found more than 50 pairs to consider."

"Here's one I made up," Ruth told the class. "It starts out with 4 different amounts of money to compare."



A. Differences.

1. $ 30.24 − 9.59 = $ 20.65
2. $ ___ − ___ = $ ___
3. $ ___ − ___ = $ ___
4. $ ___ − ___ = $ ___
5. $ ___ − ___ = $ ___
6. $ ___ − ___ = $ ___

B. Can you find 15 pairs of the differences to consider?

1. $ ___ − ___ = $ ___
2. $ ___ − ___ = $ ___
3. $ ___ − ___ = $ ___
4. $ ___ − ___ = $ ___
5. $ ___ − ___ = $ ___
6. $ ___ − ___ = $ ___
7. $ ___ − ___ = $ ___
8. $ ___ − ___ = $ ___
9. $ ___ − ___ = $ ___
10. $ ___ − ___ = $ ___
11. $ ___ − ___ = $ ___
12. $ ___ − ___ = $ ___
13. $ ___ − ___ = $ ___
14. $ ___ − ___ = $ ___
15. $ ___ − ___ = $ ___

"I notice," Ruth pointed out, "that some of the differences are repeated . . . in fact, there are 4 that are repeated. Those repeated are:

C.

1. $ 8.82
2. $ ___
3. $ ___
4. $ ___

"And, I notice that 3 of these were in the first list of 6 differences. Do you suppose this is an accident?

"In addition to the 4 repeats, there are 7 other differences.

D.

1. $ ___ − ___ = $ ___
2. $ ___ − ___ = $ ___
3. $ ___ − ___ = $ ___
4. $ ___ − ___ = $ ___
5. $ ___ − ___ = $ ___
6. $ ___ − ___ = $ ___
7. $ ___ − ___ = $ ___

"You might like to consider all pairs of these differences — the 4 repeats and the 7 others . . . 11 in all."

Arithmetic in a Caramel Company.

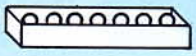
A group of boys organized a Caramel Company. They decided to package the caramels they made in this way:

7 caramels to a package.
7 packages to a box.
7 boxes to a carton.

There are 7 caramels in a package — but how many in a box, and how many in a carton?

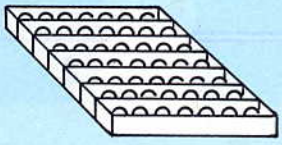
The sketches below may help.

one package:



____ caramels

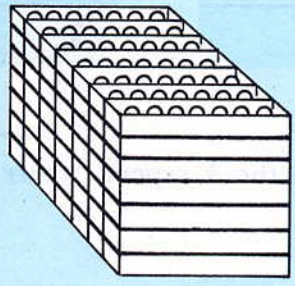
one box:



____ packages

____ caramels

one carton:



____ boxes

____ packages

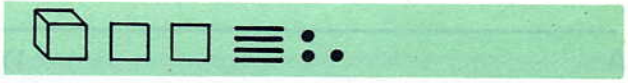
____ caramels

Tom was in charge of the packaging and shipping department.

To help keep records more easily, he invented a shorthand:

- • ..... a caramel
- — ..... a package of 7 caramels
- □ ..... a box of 7 packages
- ⧈ ..... a carton of 7 boxes

In this shorthand, he could write 1 carton, 2 boxes, 4 packages, and 3 caramels in this way:



Can you find the total number of caramels:

1 carton ..... 343 caramels

2 boxes ..... ____ caramels

4 packages ..... ____ caramels

3 caramels ..... ____ caramels

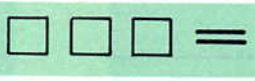
Total: ____ caramels

Here are other examples of Tom's shorthand. Please fill in the totals:

Totals

a. _____

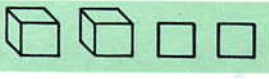
b. _____

c.  _____

d. _____

e. _____

f. _____

g.  _____

Try your hand at Tom's way of keeping records. Use the fewest containers possible.

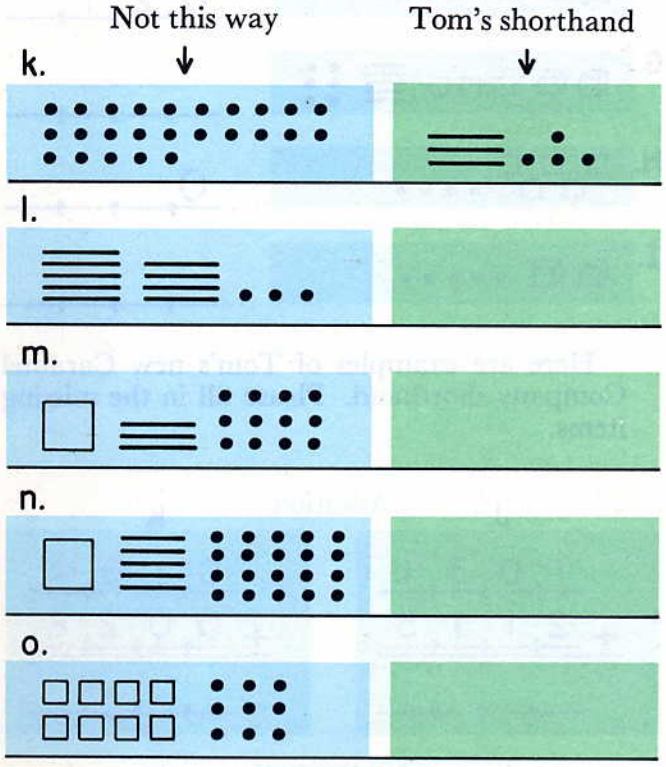
h. 8

i. 100

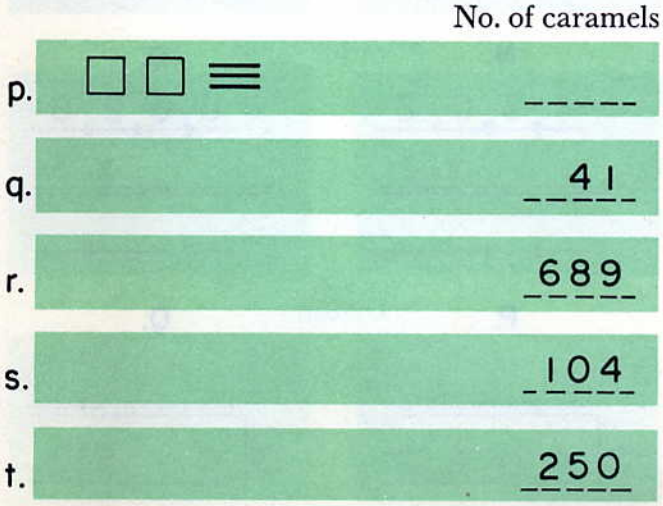
j. 59

Tom explained, "We always fill an order with the fewest cartons, boxes, packages, and separate caramels we can.

"For example: we would not ship 25 separate caramels. We would ship 3 packages and 4 loose caramels. We follow a similar rule in our shorthand."

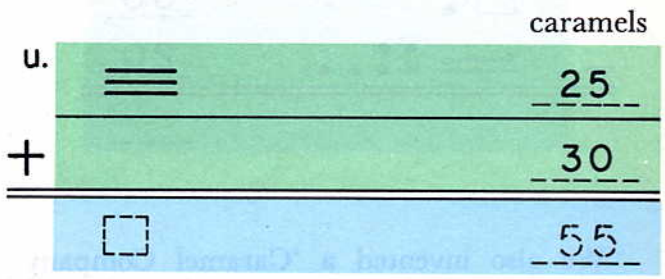


Here is some practice in reading and writing Tom's shorthand:

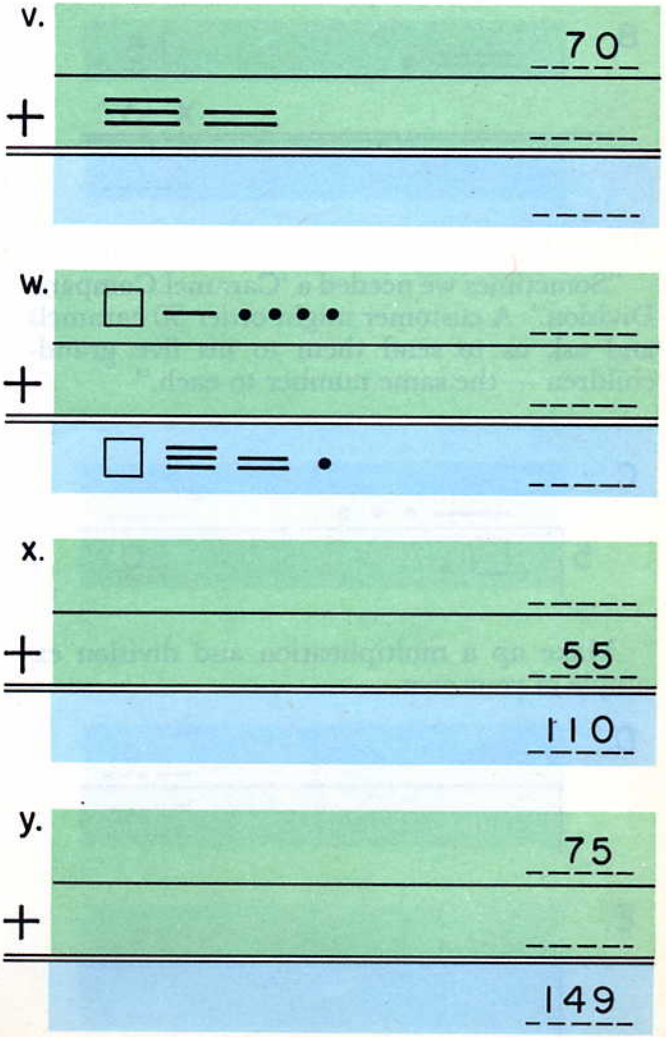


"Sometimes a customer would order some caramels — and then place another order before we had shipped the first order. So we invented a kind of addition.

"Suppose the first order was for 25 caramels and the next was for 30 caramels. Our addition looked like this:" (Please complete the examples.)



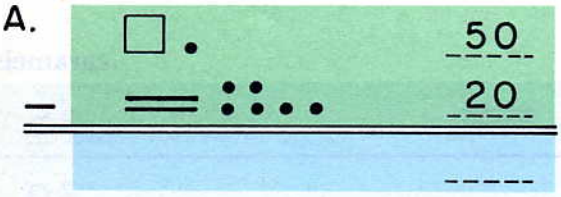
Here are other examples of Tom's "Caramel Company Addition" . . . please complete them.



"We also invented our special kind of subtraction.

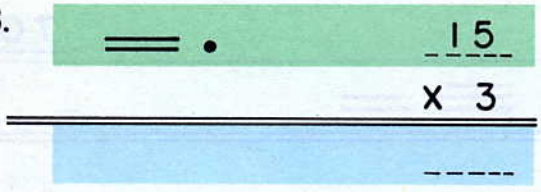
"Suppose a customer ordered 50 caramels . . . and after we had the order packed, the customer said they had ordered 20 too many. Our subtraction would look like this:

A.



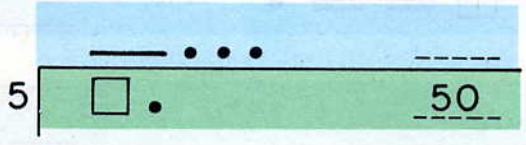
"We also invented a 'Caramel Company Multiplication.' A customer might order 15 caramels . . . then call back asking us to change his order to 3 times as many."

B.



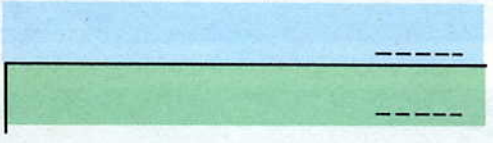
"Sometimes we needed a 'Caramel Company Division.' A customer might order 50 caramels and ask us to send them to his five grandchildren — the same number to each."

C.

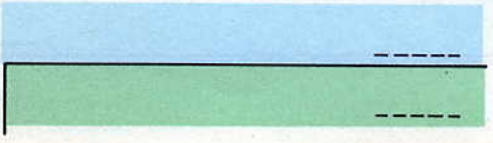


Make up a multiplication and division example of your own.

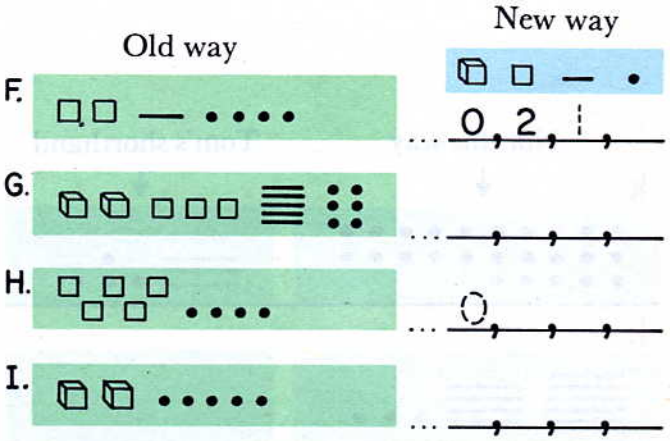
D.



E.



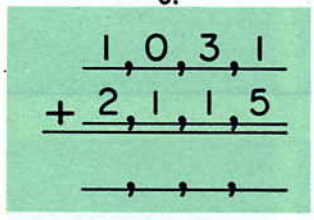
"This was a helpful shorthand," Tom explained, "but it was clumsy. So we changed to a shorthand we could write more rapidly. Here are some examples of the old and new methods. Perhaps you can fill in the blanks.



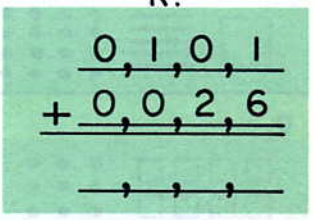
Here are examples of Tom's new Caramel Company shorthand. Please fill in the missing items.

Addition

J.

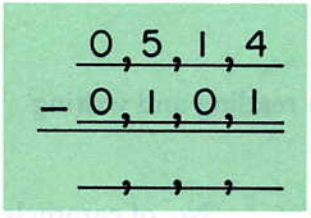


K.

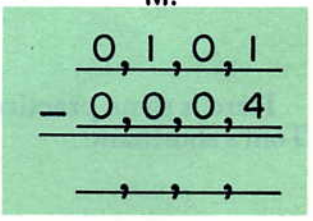


Subtraction

L.

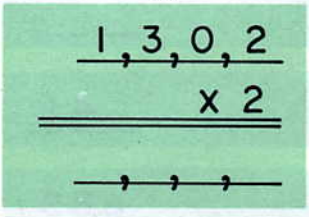


M.

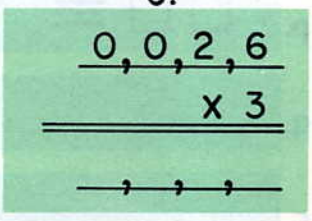


Multiplication

N.

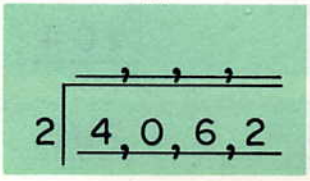


O.

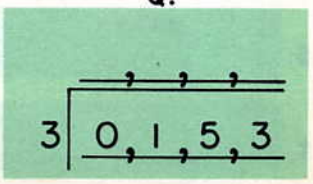


Division

P.



Q.



Here are more examples of Caramel Company arithmetic.

Remember there are ..... 7 caramels to a package.
7 packages to a box.
7 boxes to a carton.

a
$$\begin{array}{r} 1,4 \\ +\ 0,5 \\ \hline \_\_,\_ \end{array}$$

b
$$\begin{array}{r} 0,6 \\ +\ 0,6 \\ \hline \_\_,\_ \end{array}$$

c
$$\begin{array}{r} \_\_,\_ \\ +\ 1,4 \\ \hline 3,0 \end{array}$$

d
$$\begin{array}{r} 3,0 \\ -\ 0,5 \\ \hline \_\_,\_ \end{array}$$

e
$$\begin{array}{r} 5,4 \\ -\ 1,3 \\ \hline \_\_,\_ \end{array}$$

f
$$\begin{array}{r} 4,2 \\ -\ 1,4 \\ \hline \_\_,\_ \end{array}$$

g
$$\begin{array}{r} 4,5,1 \\ +\ 1,0,6 \\ \hline \_\_,\_\_,\_ \end{array}$$

h
$$\begin{array}{r} 0,4,3 \\ +\ 0,5,5 \\ \hline \_\_,\_\_,\_ \end{array}$$

i
$$\begin{array}{r} 3,0,5 \\ -\ 0,4,2 \\ \hline \_\_,\_\_,\_ \end{array}$$

j
$$\begin{array}{r} 1,0,0 \\ -\ 0,2,6 \\ \hline \_\_,\_\_,\_ \end{array}$$

1
$$\begin{array}{r} 4,3 \\ +\ 1,4 \\ \hline \_\_,\_ \end{array}$$

2
$$\begin{array}{r} 2,5 \\ +\ 2,1 \\ \hline \_\_,\_ \end{array}$$

3
$$\begin{array}{r} 1,6 \\ +\ 3,6 \\ \hline \_\_,\_ \end{array}$$

4
$$\begin{array}{r} 2,3,4 \\ +\ 0,5,1 \\ \hline \_\_,\_\_,\_ \end{array}$$

5
$$\begin{array}{r} 3,1,4 \\ +\ 1,5,4 \\ \hline \_\_,\_\_,\_ \end{array}$$

6
$$\begin{array}{r} 5,0 \\ -\ 0,2 \\ \hline \_\_,\_ \end{array}$$

7
$$\begin{array}{r} 3,6 \\ -\ 1,2 \\ \hline \_\_,\_ \end{array}$$

8
$$\begin{array}{r} 4,5 \\ -\ \_\_,\_ \\ \hline 2,6 \end{array}$$

9
$$\begin{array}{r} 3,2,5 \\ -\ 0,4,3 \\ \hline \_\_,\_\_,\_ \end{array}$$

10
$$\begin{array}{r} 5,1,2 \\ -\ \_\_,\_\_,\_ \\ \hline 4,6,3 \end{array}$$

Multiplication

11
$$\begin{array}{r} 0,3 \\ \times\ 3 \\ \hline \_\_,\_ \end{array}$$

12
$$\begin{array}{r} 3,2 \\ \times\ 2 \\ \hline \_\_,\_ \end{array}$$

13
$$\begin{array}{r} 1,5 \\ \times\ 2 \\ \hline \_\_,\_ \end{array}$$

14
$$\begin{array}{r} 1,3,2 \\ \times\ 3 \\ \hline \_\_,\_\_,\_ \end{array}$$

15
$$\begin{array}{r} 0,2,1 \\ \times\ 6 \\ \hline \_\_,\_\_,\_ \end{array}$$

16
$$\begin{array}{r} 0,4 \\ \times\ 5 \\ \hline \_\_,\_ \end{array}$$

17
$$\begin{array}{r} 1,2 \\ \times\ 4 \\ \hline \_\_,\_ \end{array}$$

18
$$\begin{array}{r} \_\_,\_ \\ \times\ 3 \\ \hline 6,3 \end{array}$$

19
$$\begin{array}{r} \_\_,\_\_,\_ \\ \times\ 2 \\ \hline 4,6,2 \end{array}$$

20
$$\begin{array}{r} \_\_,\_\_,\_ \\ \times\ 3 \\ \hline 3,5,1 \end{array}$$

21
$$\begin{array}{r} 0,6 \\ \times\ 3 \\ \hline \_\_,\_ \end{array}$$

22
$$\begin{array}{r} \_\_,\_ \\ \times\ 4 \\ \hline 6,2 \end{array}$$

23
$$\begin{array}{r} \_\_,\_ \\ \times\ 6 \\ \hline 5,1 \end{array}$$

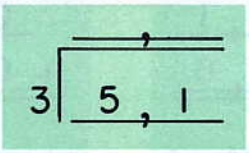
24
$$\begin{array}{r} \_\_,\_\_,\_ \\ \times\ 3 \\ \hline 1,2,6 \end{array}$$

25
$$\begin{array}{r} \_\_,\_\_,\_ \\ \times\ 2 \\ \hline 3,0,1 \end{array}$$

"I was very slow in division examples at the Caramel Company," Tom told his friends. "But we invented a method that helped a lot.

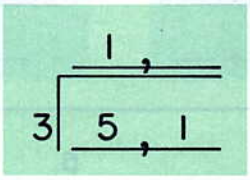
"I'll explain. We will begin with an easy example:

A1



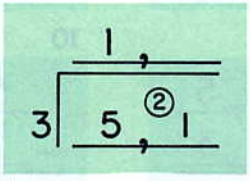
"I think — I've got 5 packages and 1 caramel to divide among 3 people. I can give each person a full package. So, I write:

A2



. . . but I have 2 full packages left over. So I write ② in the next space — next to the 1 in this example.

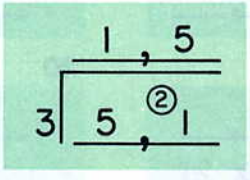
A3



"I use the circle to help me remember that it is 2 *packages* I have not yet divided.

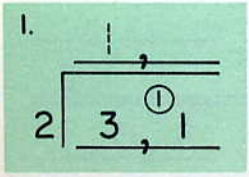
"Since there are 14 caramels in 2 packages and there is another caramel — 15 in all — I give 5 to each of the three people and write:

A4

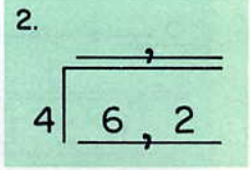


Here are two more examples. Please try to complete them yourself using Tom's method.

1.

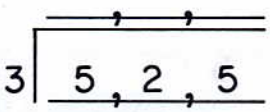


2.



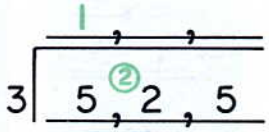
Harder examples follow a similar pattern. Here is one:

B1



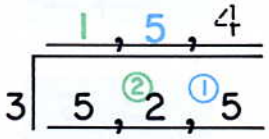
"I think — 5 boxes, 2 packages, and five caramels to divide among 3 people. I begin by giving each a box — and note that I have 2 of the 5 boxes left.

B2



"I think of the ② boxes left over as 14 packages. These added to the two I started with make 16 packages. I can give 5 packages to each of the 3 people — and have 1 package left. So I write:

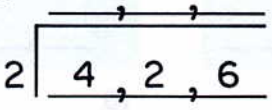
B3



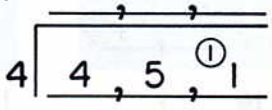
"Finally, ① package plus 5 caramels is 12 caramels — 4 to each person."

Here are more examples for you to try, using Tom's method.

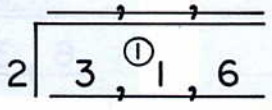
3.



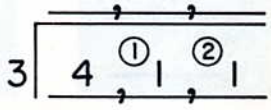
4.



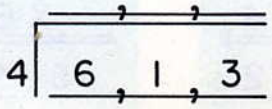
5.



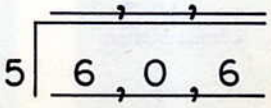
6.



7.



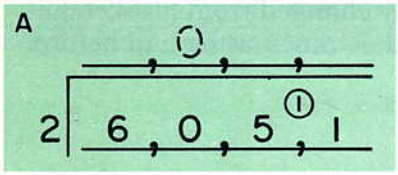
8.



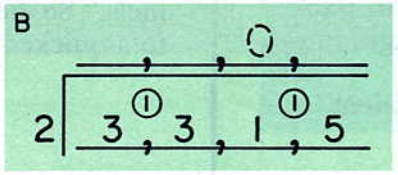
Tom used his methods in problems involving cartons as well as boxes, packages, and separate caramels.

Tom explained, "I can take care of the cartons first, then the boxes, then the packages, and finally — the loose caramels."

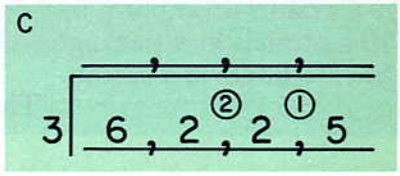
A



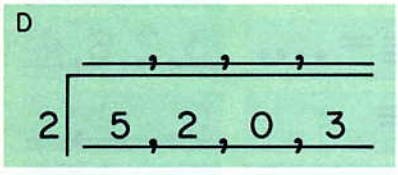
B



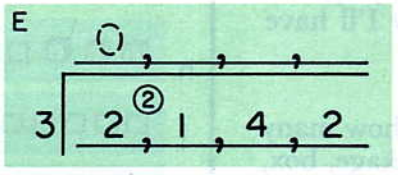
C



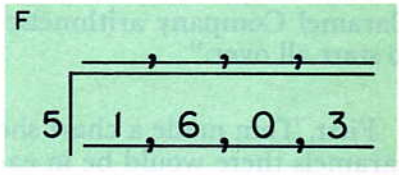
D



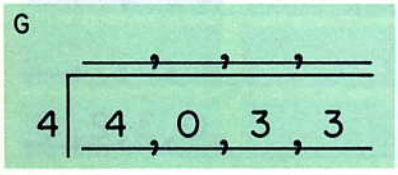
E



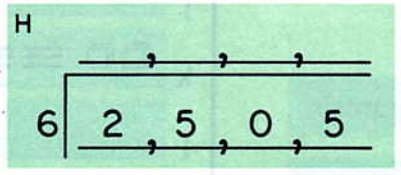
F



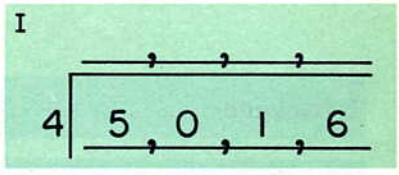
G



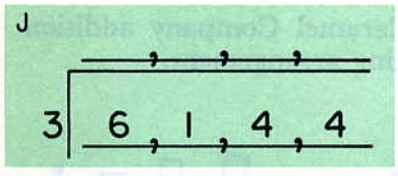
H



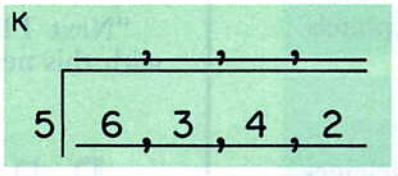
I



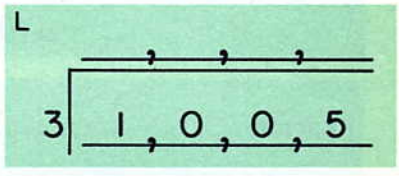
J



K

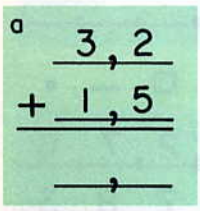


L

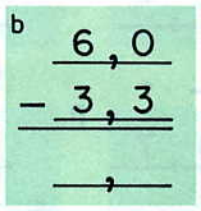


Here is a test Tom made up using Caramel Company arithmetic.

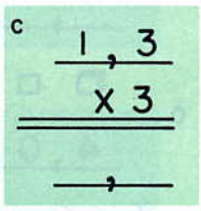
a



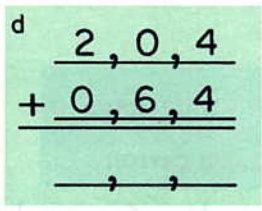
b



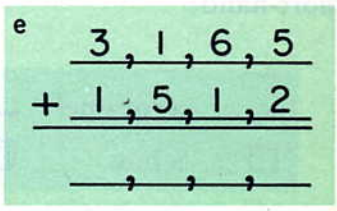
c



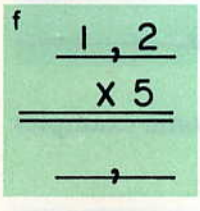
d



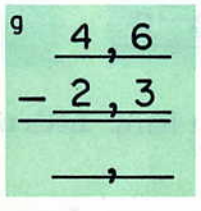
e



f



g



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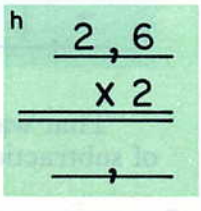

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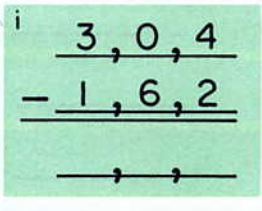

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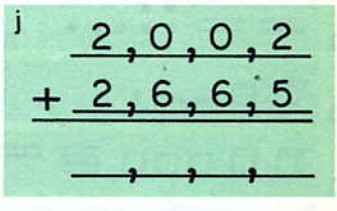

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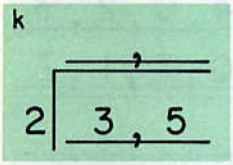

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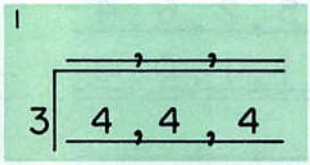

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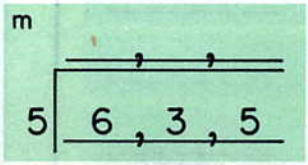

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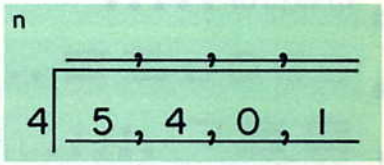


Tom Gets Shocking News!..... (continued on page 30)

Tom came home from vacation one summer and found the following notice on the bulletin board:

IMPORTANT

Caramel Company has decided to pack 10 caramels in a package, 10 packages in a box, 10 boxes in a carton.

The President.

Tom said, "Oh well, there goes my special Caramel Company arithmetic! Now I'll have to start all over."

First, Tom made a chart showing how many caramels there would be in each package, box, and carton.

1 package	10 caramels
1 box	10 packages 100 caramels
1 carton	_____ boxes _____ packages _____ caramels

Tom experimented with his old sketch-short-hand:

•..... a caramel	—.... a package
□.... a box	⧉..... a carton

		No. of caramels
a.	═ ═ ⁘	_____
b.	⧉ □ — •	_____
c.	⧉ ⧉ □□□ ≡ ═	_____
d.	□□□□ ⁙	_____
e.	⧉ ⧉ ⧉ □ ≡ ≡ ••	_____
f.	≡ ≡ ¯ ⁙	_____
g.		326

A smile appeared on Tom's face. "Ahah! I'm beginning to like this new packing scheme.

"There is one big trouble," Tom noted. "Sometimes I need to sketch as many as 9 containers of the same kind or 9 separate caramels." So, he quickly changed from his sketches to a quicker method — much as he had before.

		⧉ □ — •
h.	⧉ ⧉ ⧉ □□□ ≡ ═ ••••	3,2,6,4
i.	□□□□□ ≡ ≡	0, , ,
j.	⧉ ⧉ ⧉ ⧉ ⧉ ⧉ □	, , ,
k.	⧉ ⧉ ≡ ≡ ≡	, , ,
l.		0,3,2,5

"Next I'll try Caramel Company addition with this new packing arrangement.

	⧉ □ — •	⧉ □ — •
m.	1,0,0,6 + 3,0,0,7 , , , 3	
n.		1,8,6,9 + 0,6,0,1 , , ,
o.	4,0,9,7 + , , , 7,1,0,2	
p.		3,2,7,6 + 1,7,8,3 , , ,

"That wasn't hard. Let's try some examples of subtraction."

q.	5,2,6,4 − 1,1,0,8 , , ,	
r.		4,0,2,3 − 0,7,1,7 , , ,

Try some examples of your own.

Here are some more examples of Caramel Company Subtraction.

1
$$\begin{array}{r} 0,4,2,3 \\ -\ 0,2,0,7 \\ \hline \end{array}$$

2
$$\begin{array}{r} 7,4,6,0 \\ -\ 0,7,5,8 \\ \hline \end{array}$$

3
$$\begin{array}{r} 8,3,2,9 \\ -\ 4,1,5,3 \\ \hline \end{array}$$

4
$$\begin{array}{r} 0,1,0,0 \\ -\ 0,0,3,6 \\ \hline \end{array}$$

5
$$\begin{array}{r} 2,3,5,9 \\ -\ 0,3,7,3 \\ \hline \end{array}$$

6
$$\begin{array}{r} 6,3,0,0 \\ -\ 3,1,5,3 \\ \hline \end{array}$$

7
$$\begin{array}{r} 7,0,0,0 \\ -\ 4,1,2,5 \\ \hline \end{array}$$

8
$$\begin{array}{r} 9,3,1,6 \\ -\ \\ \hline 4,5,5,8 \end{array}$$

Here are some examples of Caramel Company Multiplication.

9
$$\begin{array}{r} 2,3,0,1 \\ \times\ 3 \\ \hline \end{array}$$

10
$$\begin{array}{r} 0,9,0,7 \\ \times\ 2 \\ \hline \end{array}$$

11
$$\begin{array}{r} 1,5,2,5 \\ \times\ 3 \\ \hline \end{array}$$

12
$$\begin{array}{r} 2,1,3,2 \\ \times\ 4 \\ \hline \end{array}$$

13
$$\begin{array}{r} 0,6,4,2 \\ \times\ 5 \\ \hline \end{array}$$

14
$$\begin{array}{r} 0,4,3,1 \\ \times\ 8 \\ \hline \end{array}$$

15
$$\begin{array}{r} 1,4,6,3 \\ \times\ 6 \\ \hline \end{array}$$

16
$$\begin{array}{r} 0,8,4,3 \\ \times\ 7 \\ \hline \end{array}$$

17
$$\begin{array}{r} 0,2,3,1 \\ \times\ 7 \\ \hline \end{array}$$

18
$$\begin{array}{r} 0,2,3,1 \\ \times\ 10 \\ \hline \end{array}$$

19
$$\begin{array}{r} 0,2,3,1 \\ \times\ 17 \\ \hline \end{array}$$

20
$$\begin{array}{r} 0,2,3,1 \\ \times\ 27 \\ \hline \end{array}$$

Watch the Signs!

21
$$\begin{array}{r} 3,7,2,5 \\ +\ 4,8,4,7 \\ \hline \end{array}$$

22
$$\begin{array}{r} 4,3,6,9 \\ -\ \\ \hline 2,4,8,2 \end{array}$$

23
$$\begin{array}{r} \\ \times\ 4 \\ \hline 6,8,5,2 \end{array}$$

24
$$\begin{array}{r} \\ -\ 5,2,9,3 \\ \hline 2,0,5,3 \end{array}$$

25
$$\begin{array}{r} \\ \times\ 3 \\ \hline 5,1,0,0 \end{array}$$

26
$$\begin{array}{r} 3,7,4,6 \\ +\ \\ \hline 9,9,0,1 \end{array}$$

27
$$\begin{array}{r} 9,4,1,6 \\ -\ 6,7,8,9 \\ \hline \end{array}$$

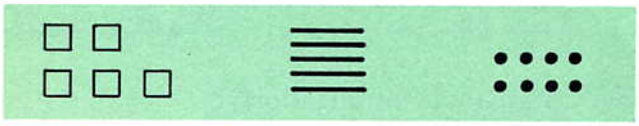
28
$$\begin{array}{r} 5,2,8,4 \\ -\ \\ \hline 4,3,7,5 \end{array}$$

Caramel Company Division.

"We adapted our Caramel Company way of doing division to the new packing system," Tom explained.

"Here is an example.

"Suppose I have 5 boxes, 5 packages, and 8 loose caramels and I want to divide them into 3 piles with the same number of caramels in each pile.

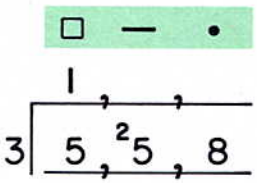


5 boxes (500 caramels) | 5 packages (50 caramels) | 8 loose caramels (8 caramels)

"That's a total of _____ caramels."

Step One.

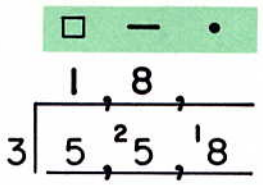
"First, I can put one of the boxes in each new pile . . . leaving 2 boxes I must change into packages. I record this first step this way:



"The two boxes can be opened and become 20 packages to add to the 5 I already have . . . a total of 25. By writing the '2' as I show it, it serves a double purpose. It shows how many boxes I take the packages out of . . . and writing it to the left of the '5' is a quick way of noting 25 packages."

Step Two.

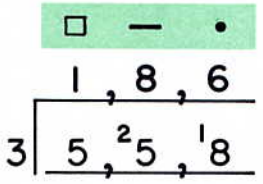
"Of the 25 packages, I can put 8 in each of the three piles, leaving 1 package undivided. I record that step this way."



Step Three.

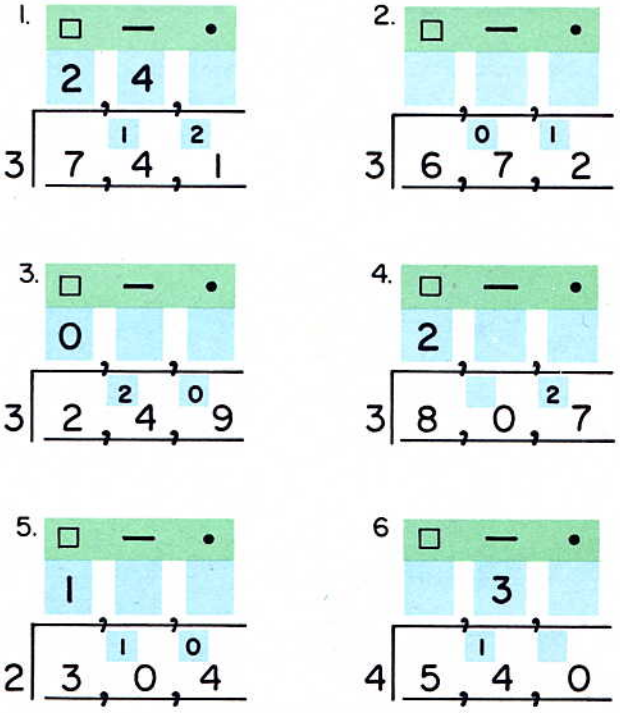
"There isn't much left to do. I open the package of 10 caramels, add them to the 8 loose ones . . . for a total of 18 loose caramels.

"I put 6 caramels in each pile and I'm done!



"I now have 1 box, 8 packages, and 6 loose caramels in each pile.

"Here are some examples with little blue blocks that indicate my work. I'll leave some of the blue blocks blank . . . for you to fill in according to my system."



More examples of:

Caramel Company Division.

(We will omit the marks that indicate boxes, packages, and loose caramels.)

a. 5 | 825

b. 6 | 2,558

c. 8 | 8,224

d. 7 | 2,455

"We'll omit the large blue blocks."

e. 9 | 2,888

f. 2 | 9,338

g. 3 | 2,004

h. 5 | 7,115

i. 4 | 1,000

j. 6 | 6,330

"Of course," Tom pointed out, "this system can handle problems including cartons of caramels.

And, when we had to handle 'skids' with 10 cartons per skid, our system could help us find results."

k. 3 | 4,275

l. 5 | 7,135

m. (a skid) 4 | 5,047,2

We can begin omitting the large blue blocks.

n. 5 | 2,340

o. 6 | 8,514

p. 4 | 6,112,8

Finally, we can begin omitting the small blue blocks.

q. 7 | 2,543,14

r. 8 | 3,224

s. 5 | 7,184,5

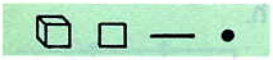
t. 9 | 9,378

u. 6 | 9,294

v. 4 | 9,807,6

Betty's Discovery.

"Tom!" Betty exploded. "Your new caramel packing scheme is a lot like the way we write down amounts of money."



"A dime is like a package of 10 pennies.

"A dollar is like a box of 10 dimes.

"A ten-dollar bill is like a carton of 10 dollars."

Betty's Guess.

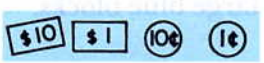
"And, I'll bet you can use your Caramel Company methods to add and subtract amounts of money and to multiply and divide them by numbers.

"Let's try!"

Everyone helped to find out whether Betty's guess was a good one.

Here are some of the examples they tried. Please fill in the blanks.

ADDITION.



a
7,534
+ 0,608
1,142

b
3,284
+ 2,792

c
4,372
+ 0706

d
0,586
+ 0586

e
0,307
+ 1,993

f
1,725
+ 1,725

SUBTRACTION.

g
0,042
− 0,007
5

h
0,525
− 0,250

i
1,450
− 0,829

j
4,325
− 1,835

You make up two examples.

k
−

l
−

MULTIPLICATION.

m
0,125
× 3

n
1,081
× 7

o
1,235
× 4

p
1,538
× 6

You make up two examples.

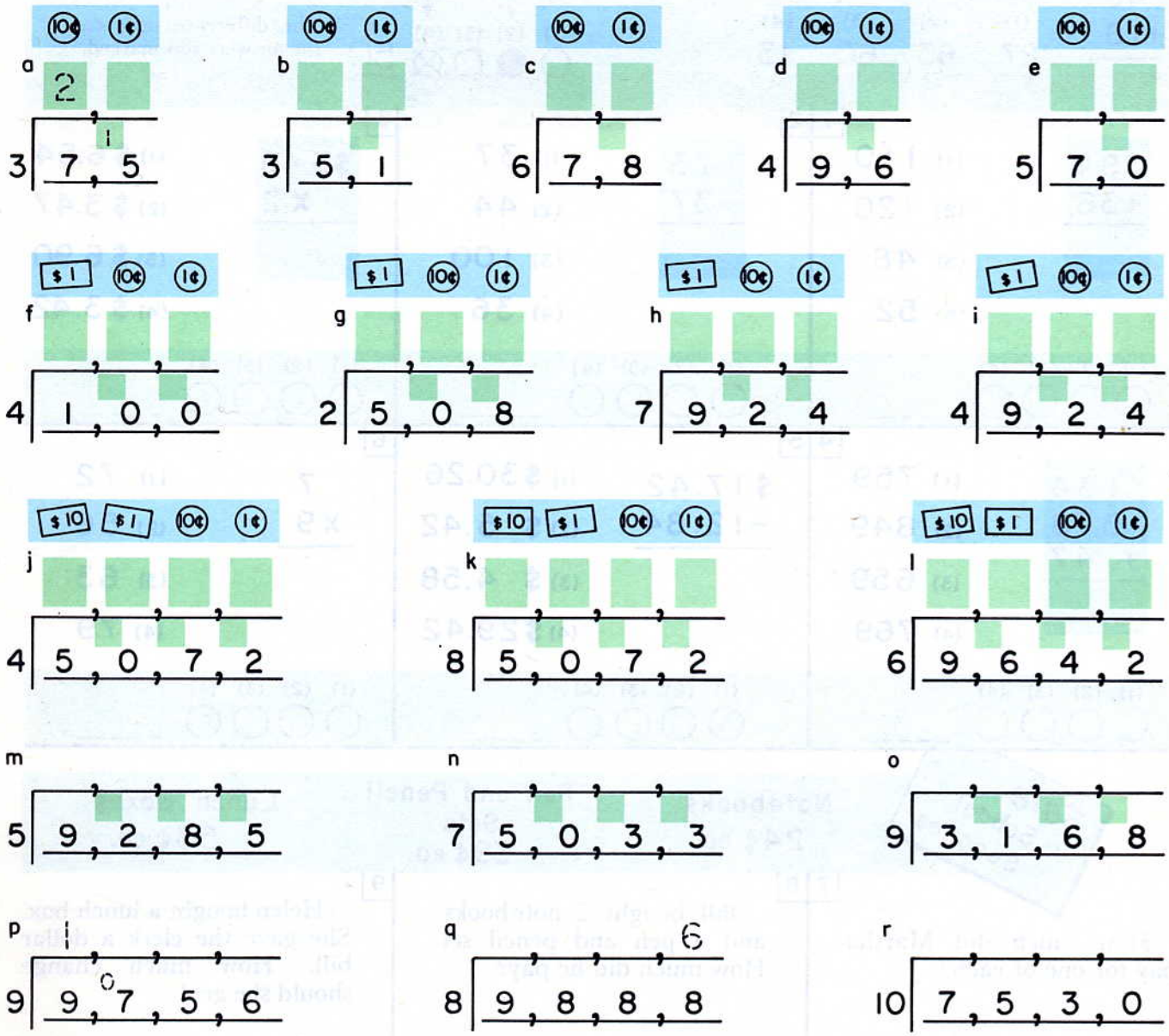
q
×

r
×

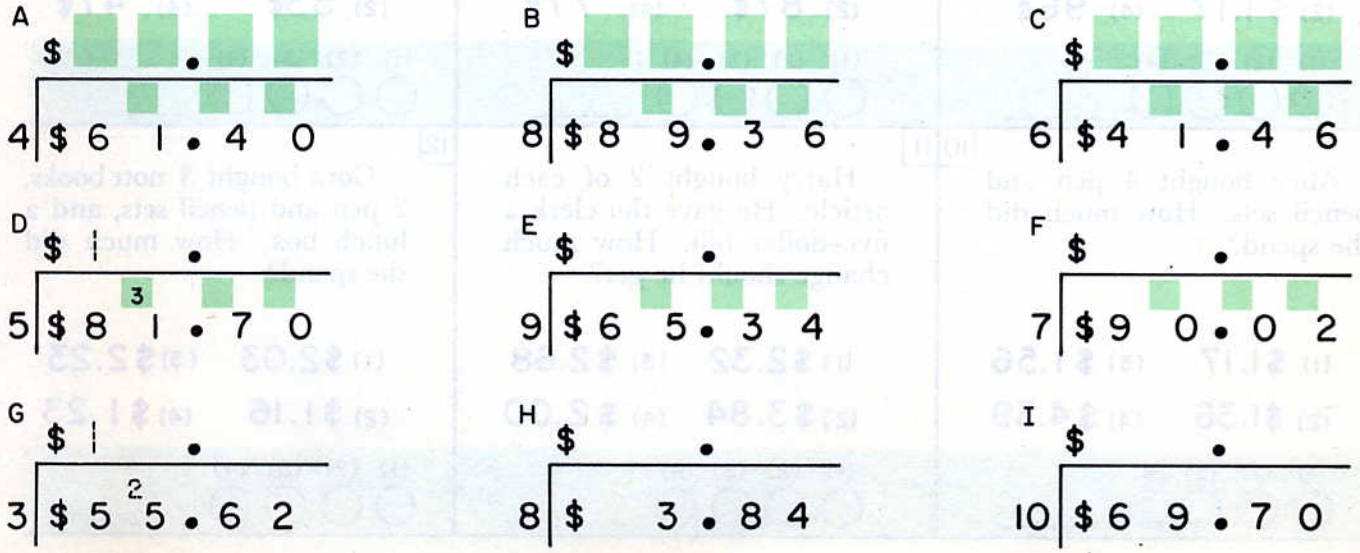
"I thought I was right!" Betty was excited.

"And I think that kind of division you showed us will work with amounts of money."

DIVISION.



Betty experiments with her own way of writing amounts of money.



Following Directions.

Mark the correct answer. Mark the furthest from the correct answer. The difference between the answers you marked.

45 +18 (1) 27 (2) 63 (3) 60 (4) 13

(1) (2) (3) (4) 50

1 | 2

84 +36

(1) 110
(2) 120
(3) 48
(4) 52

(1) (2) (3) (4) _____

73 −37

(1) 37
(2) 44
(3) 100
(4) 36

(1) (2) (3) (4) _____

3

$3.45 X 2

(1) $6.54
(2) $3.47
(3) $6.90
(4) $3.42

(1) (2) (3) (4) _____

4 | 5

134 578 + 47

(1) 759
(2) 849
(3) 659
(4) 769

(1) (2) (3) (4) _____

$17.42 −12.84

(1) $30.26
(2) $ 5.42
(3) $ 4.58
(4) $29.42

(1) (2) (3) (4) _____

6

7 X9

(1) 72
(2) 56
(3) 63
(4) 79

(1) (2) (3) (4) _____

Sale School Supplies

Notebooks 24¢ ea.

Pen and Pencil Sets 39¢ ea.

Lunch Boxes 53¢ ea.

7 | 8

How much did Martha pay for one of each?

(1) $1.16 (3) 63¢
(2) $1.17 (4) 96¢

(1) (2) (3) (4) _____

Bill bought 2 notebooks and a pen and pencil set. How much did he pay?

(1) 63¢ (3) $1.02
(2) 87¢ (4) 77¢

(1) (2) (3) (4) _____

9

Helen bought a lunch box. She gave the clerk a dollar bill. How much change should she get?

(1) 57¢ (3) $1.53
(2) 53¢ (4) 47¢

(1) (2) (3) (4) _____

10 | 11

Alice bought 4 pen and pencil sets. How much did she spend?

(1) $1.17 (3) $1.56
(2) $1.36 (4) $4.39

(1) (2) (3) (4) _____

Harry bought 2 of each article. He gave the clerk a five-dollar bill. How much change should he get?

(1) $2.32 (3) $2.68
(2) $3.84 (4) $2.00

(1) (2) (3) (4) _____

12

Cora bought 3 notebooks, 2 pen and pencil sets, and a lunch box. How much did she spend?

(1) $2.03 (3) $2.23
(2) $1.16 (4) $1.23

(1) (2) (3) (4) _____

Show the largest amount with an A in the green block; the next largest with a B; the next with a C; etc. Write the difference in each example.

I

3 yards, 2 feet, 7 inches .....

100 inches .....

12 feet .....

Difference between

A and B: _____ inches

B and C: _____ inches

A and C: _____ inches

II

15 dimes and 40 nickels .....

40 dimes and 230 pennies .....

57 dimes .....

Difference between

A and B: _____ ¢

B and C: _____ ¢

A and C: _____ ¢

III

37 hours .....

1 day and 5 hours .....

600 minutes .....

Half a day .....

Difference between

A and B: ____ hrs. | B and D: ____ hrs.

B and C: ____ hrs. | C and D: ____ hrs.

A and C: ____ hrs. | A and D: ____ hrs.

IV

An hour and a quarter .....

240 minutes .....

Three half-hours .....

1 hour and 90 minutes .....

Difference between

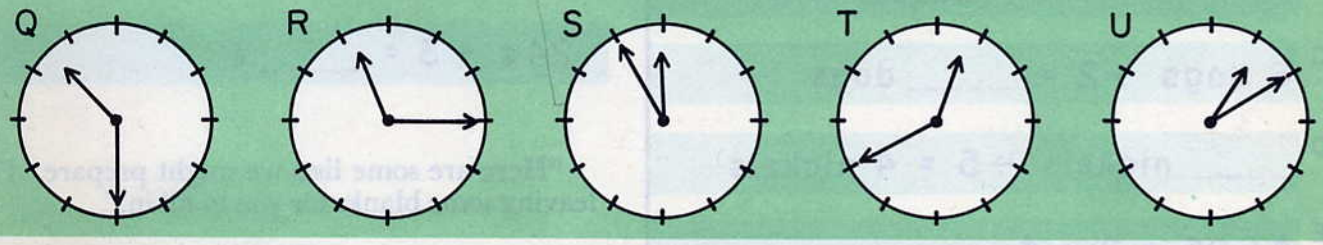
A and B: _____ B and D: _____

B and C: _____ C and D: _____

A and C: _____ A and D: _____

V

The game was to start at 1:15 P.M. Bill looked at the clock 5 different times before the game started.



How long was it before game time in each case?

Q : _____ min.

R : _____ min.

S : _____ min.

T : _____ min.

U : _____ min.

How long was it between the times he looked at the clock?

Q and R: _____ min. S and U: _____ min.

R and S: _____ min. R and T: _____ min.

S and T: _____ min. Q and U: _____ min.

T and U: _____ min. Q and S: _____ min.

Some classmates asked Ben and Ruth about multiplication and division in the "Arithmetic of Things and Numbers."

Jerry asked, "What would you do if your Mother gave you this problem?"

3 dogs x 2 dogs = _____

or

4 nickels x 5 dimes = _____

"Mother never gave us problems like those," Ruth told the class.

"Ben and I couldn't have made any sense out of such questions.

"In the arithmetic Mother taught us, we never multiplied things by things; rather, we multiplied things by numbers . . . and we divided things by numbers. We would have problems such as:

A 3 dogs x 2 = 6 dogs

B 4 nickels x 5 = _____ nickels

C 6 dogs ÷ 2 = _____ dogs

D _____ nickels ÷ 5 = 4 nickels

E $1.25 x 2 = $ _____

F 8 in. x 3 = _____ feet

G 7 in. x 4 = _____ ft., _____ in.

H 2 hrs. ÷ 3 = _____ minutes

I 1 day ÷ 6 = _____ hours

"But Ben and I considered such problems too easy. We would often answer the question asked — and several more at the same time.

"For example, Mother might say:

7 stars x 4 = _____ stars

"We would prepare lists like this. You can fill in the blanks."

J

	Numbers of stars						
	1	3	4	7	10	17	
x 4	4						

or

K

	Numbers of stars						
	2	5	7	12	19	31	
x 4							

or

L

	Numbers of stars						
	8	9	10	18	19	20	
x 4							

"Or, Mother might say:

24¢ ÷ 3 = _____ ¢

"Here are some lists we might prepare. I'm leaving some blanks for you to fill in."

M

	Numbers of cents						
	12	24	36	60			
÷ 3					32	52	

N

	Numbers of cents						
	24			108		120	
÷ 3		10	20		30		

"Sometimes Ben and I would give Mother answers that showed her we knew several ways to tackle hard problems. Here are several examples — with plenty of blanks for you."

A. 9 days x 7 = _____ days

Numbers of days

1.

	6	3	9
x 2	12		
x 5	30		
x 7			

2.

	4	5	9
x 3			
x 4			
x 7			

3.

	3	6	9
x 1			
x 6			
x 7			

B. 16 pounds x 8 = _____ pounds

Numbers of pounds

4.

	9	7	16
x 3			
x 5			
x 8			

5.

	11	5	16
x 4			
x 4			
x 8			

6.

	6	10	16
x 1			
x 7			
x 8			

C. 13 miles x 14 = _____ miles

Numbers of miles

7.

	6	7	13
x 8			
x 6			
x 14			

8.

	9	4	13
x 7			
x 7			
x 14			

9.

	3	10	13
x 4			
x 10			
x 14			

D. 15¢ x 15 = _____ ¢

Numbers of cents

10.

	6	9	15
x 8			
x 7			
x 15			

11.

	2	13	15
x 1			
x 14			
x 15			

12.

	10	5	15
x 5			
x 10			
x 15			

"Mother was especially pleased when our list of results showed we were looking for unusual shortcuts."

A. 9 gallons x 8 = _____ gallons

Numbers of gallons

1.

	10	1	9
x 10			
x 2			
x 8			

2.

	4	5	9
x 4			
x 4			
x 8			

3.

	3	6	9
x 3			
x 5			
x 8			

B. 15 feet x 7 = _____ feet

Numbers of feet

4.

	8	7	15
x 3			
x 4			
x 7			

5.

	20	5	15
x 2			
x 5			
x 7			

6.

	10	5	15
x 10			
x 3			
x 7			

C. 19 pounds x 18 = _____ pounds

Numbers of pounds

7.

	11	8	19
x 9			
x 9			
x 18			

8.

	20	1	19
x 20			
x 2			
x 18			

9.

	10	9	19
x 10			
x 8			
x 18			

D. $23 x 45 = $ _____

Numbers of dollars

10.

	11	12	23
x 15			
x 30			
x 45			

11.

	20	3	23
x 50			
x 5			
x 45			

12.

	3	20	23
x 5			
x 40			
x 45			

"Mother taught us to do some of our work in our heads. She said that neatness was important and showed us how to write our work in little squares like this:

A 26 pounds x 17 = _____ lbs.

	20	6	26
x 7			
x 10			
x 17			

		2	6
	x	1	7
	1	8	2
	2	6	0
			2

B 39¢ x 41 = $ _____

		9	39
1			
x41			

		3	9
	x	4	1

C 28 tiles x53 = _____ tiles

			28
x53			

		2	8
	x		

D 2 ft. and 8 in. x 39 = _____ inches

Number of inches

	30	2	
x 39			

	x		

"For practice in neatness, she would leave some of the lines out. Soon, we didn't need any lines at all."

E

		3	6
	x	1	4

F

		1	9
	x	5	2

G

		2	8
	x	2	3

H

		7	4
	x	1	9

I

		5	1
	x	4	7

J

		9	2
	x	2	9

K

		6	6
	x	8	6

L

		4	8
	x	7	7

M

		8	7
	x	4	0

N

		5	0
	x	1	8

"When some of the numbers were larger, we had to expand our charts. We also did some mental arithmetic and recorded the results to the right of the charts."

A. 164 bushels x 13 = _____ bu.

	100	60	4	164
x 3				
x 10				
x 13				

			1	6	4
			x	1	3

B. $247 x 25 = $ _____

	200		7	247
x 5				
x 20				
x 25				

			x	2	5

C. 508 pounds x 124 = _____ lbs.

		0	8	508
x 4				
x 20				
x 100				
x 124				

			5	0	8
		x	1	2	4

D. 167 feet x 218 = _____ feet

				167
x 218				

		x			

"Mother liked to make puzzles out of practice examples. Her rules were simple: Complete each example by writing the correct digit in each empty box. Write nothing outside the boxes."

1.
```
  5
x 3
---
  1 5
```

2.
```
  3
x
---
    8
```

3.
```
  9
x
---
2
```

4.
```
  
x 6
---
2
```

5.
```
  7
x
---
    9
```

6.
```
  2
x 4
---
1 0 0
```

7.
```
  
x 9
---
1 0
```

8.
```
  
x
---
2 5
```

9.
```
  
x 8
---
5
```

10.
```
  
x 9
---
    1
```

11.
```
1 0
x
---
7
```

12.
```
  9
x
---
    3
```

13.
```
  3
x
---
1 5 0
```

14.
```
1   1
  x 5
---
9 0
```

15.
```
  8
x
---
3
```

16.
```
  
x
---
6 4
```

17.
```
  6
x
---
4 2
```

18.
```
  
x
---
1 3
```

19.
```
  
x 8
---
7
```

20.
```
    7
x
---
1 1 9
```

21.
```
  
x
---
1 1 5
```

22.
```
  5
x
---
2 0 0
```

23.
```
    3
x
---
2 1 2
```

24.
```
4
  x
---
9 2 2
```

25.
```
    5
x
---
1 7 0
```

26.
```
  
x 6
---
5 4 0
```

27.
```
  
x 2
---
2 4 2
```

28.
```
  7 0
x
---
  9
```

29.
```
  
x 9
---
  8 7
```

30.
```
  9
x 5
---
4 9 5
```

31.
```
  
x 8
---
6 4 0
```

32.
```
1 1 8
x
---
  9 0
```

33.
```
  
x
---
5 0 7
```

34.
```
  1 3
x
---
    6
---
1 5
```

35.
```
  
x 1 2
---
2
---
  1 2
```

36.
```
  
x
---
1 0 0
2 5
---
```

37.
```
  
x   5
---
  7 5
---
2   5
```

38.
```
  
x
---
---
1 2 1
```

39.
```
    3
x 1
---
---
  9 9
```

40.
```
    
  x 1 1
---
1 2 4
---
1 3 6 4
```

41.
```
  2
x   7
---
    4
---
5 5 8 9
```

42.
```
  
x
---
  2 5 5
  8 5
---
```

43.
```
    
  x   5
---
5 6 0
---
6 0 0 0
```

44.
```
    4 3
x
---
---
1 8 4
```

"Of course, we sometimes gave Mother long lists that included the answer to the problem she gave us. Here's an example."

12 books x 15 = _____books

(A)

Numbers of books

	2	3	7	10	12	20	22		27		37
x 5								115		150	

Numbers of books

	2	3	7	10	12	20	22	23	27	30	
x 10											

Numbers of books

	2	3	7	10	12	20	22	23			
x 15											

"We learned to save a lot of space in making such lists of answers. I'll give you two examples. Mother might say:

16 tiles x 13 = _____ tiles

(B)

Numbers of tiles

	2	4	6	10	16	20	30	5	23	25	48
x 3											
x 10											
x 13											

17 seats x 14 = _____ seats

(C)

Numbers of seats

	3	4	7	10	20	30	17	13	24	37	50
x 5											
x 9											
x 14											
x 4											
x 10											

A. $75 x 12 = $ _____

	10	20	30	40	70	5	75	150
x 2								
x 10								
x 12								
x 4								
x 8								

B. $175 x 27 = $ _____

	25	50	100	75	150	175
x 7						
x 20						
x 27						

C. 87 sq. ft. x 96 = _____ sq. ft.

	100	10	40	80	7	87
x 6						
x 90						
x 96						

D. $105 x 132 = $ _____

	5	100	105	210	315
x 2					
x 30					
x 100					
x 132					

Mrs. Brown Raises a Question.

"I'm very happy to learn about your Arithmetic of Things and Numbers," Mrs. Brown told Ben and Ruth. "In fact, the whole class has been interested.

"As you have been explaining, I noted something I'd like to discuss.

"You say:

3 apples + 4 apples = 7 apples

3 hours + 4 hours = 7 hours

3 trucks + 4 trucks = 7 trucks

"Why can't we simply say:

3 + 4 = 7

meaning, 3 things + 4 things equals 7 things?"

Ben thought a minute. He looked at Ruth, then asked:

"But what about this?"

3 dimes + 4 nickels = _____

Jennie said, "Well, we could say that 3 dimes plus 4 nickels equals 7 coins."

"That's right," Ben agreed, "but we could also say the result was any of the following:

_____ pennies

or _____ nickels

or _____ dimes

or _____ quarters

or _____ half-dollar

"It seems to me," Wanda said, "that most of the time there isn't any question what things we are talking about. So when we write:

3 + 4 = 7

we assume we are talking about the same kind of things."

Al joined the discussion. "I agree with Wanda. If there might be any question, then you ought to say exactly what it is you are talking about."

"I think they are right," Ruth admitted to Ben. "Mother always said that what we say in arithmetic ought to make sense. When we mean something special, we ought to make our meaning clear. If I write:

3 + 4 = 1

people will not understand. They will say I am wrong unless I explain that I mean:

3 days + 4 days = 1 week.

Now everyone would agree with my result."

"Perhaps we could agree," Mrs. Brown suggested, "that we will always indicate a special situation where it wouldn't otherwise be clear. Our motto can be: When in doubt, explain!"

Ben still was not happy. "How about multiplication? In addition:

Things + Things = Things.

"But in multiplication:

Things x a Number = Things

"Here are some examples:

3 days	x 4 =	1 week and 5 days
3 inches	x 4 =	1 foot
3 bags	x 4 =	12 bags

"Don't we need to know what we are multiplying?" Ben asked.

Tom joined in. "But there is something the same about Ben's three examples. I could write them this way:

3 days	x 4 =	12 days
3 inches	x 4 =	12 inches
3 bags	x 4 =	12 bags

and the idea they all contain is what I mean when I write:

$$3 \times 4 = 12$$

"Isn't that reasonable, Ben?"

"Not yet. When you write it as you have, how do I know whether the 3 refers to things or to a number to multiply by? You might be thinking of:

3 days x 4 = ____

or

3 x 4 days = ____

3 days multiplied by 4, or 3 times 4 days."

Tom had his answer ready: "But the answer is 12 days or 1 week and 5 days either way."

"You are right," Ben had to admit, "but I'm still not too happy. When I see:

$$6 \times 4 = 24$$

I don't know whether you started with 6 things or with 4 things."

"But you do know," Tom argues, "that the result is 24 things in either case."

"I see Ben's point," Mary said. "I think when we want people to remember what things we are talking about, we should tell them. Otherwise we don't need to write out the whole story."

Mrs. Brown had an idea. "I'm going to write down some questions in arithmetic. I'll not tell you what things I'm thinking about, nor will I tell you when I am thinking about numbers or when I'm thinking about things. How do you think I would complete each of the following statements?"

A. $17 + 5 =$ _____ B. $8 \times 3 =$ _____

C. $16 \div 2 =$ _____ D. $30 - 9 =$ _____

E. $\begin{array}{r} 13 \\ \times\ 3 \\ \hline \end{array}$ F. $\begin{array}{r} 37 \\ +28 \\ \hline \end{array}$ G. $\begin{array}{r} 25 \\ \times\ 3 \\ \hline \end{array}$

H. $4\overline{)24}$

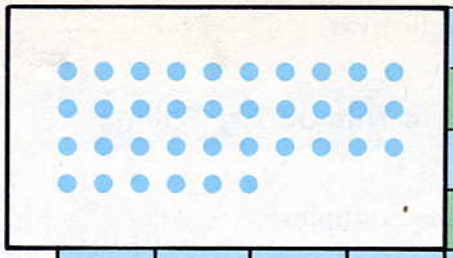
"Suppose I had been thinking of marbles, or inches, or elephants, or pennies. Would you consider that your correct answers would make sense?"

An Experiment.

Mrs. Brown prepared the material on the next 4 pages and explained to the class:

"Here is a strange mixture of numbers and things. Please fill in the blanks. If there is anything confusing, we'll talk about it."

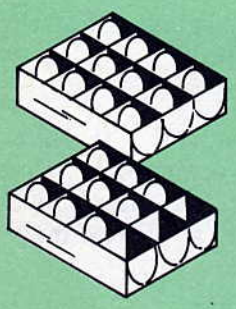
A



x 2	÷ 2	+ 15	÷ 4	− 29	+ 29

+ 4	− 10
40	
+ 16	+ 63

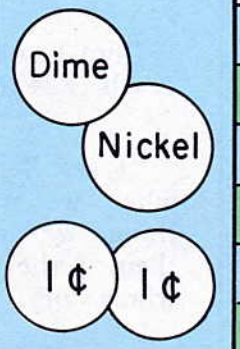
B



x 2	+ 19	÷ 3

+ 2
23
− 17
− 20

C



x 2	x 4	+ 95¢
¢	¢	¢

+ 8¢
¢
+ 26¢
¢
x 3
¢

D

no. of dozens			
÷ 12	+ 57	− 75	− 97

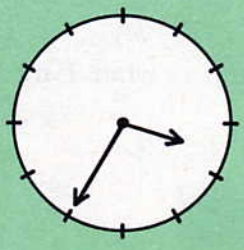
÷ 2	− 44
x 1	x 2
+ 16	− 16

E

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	

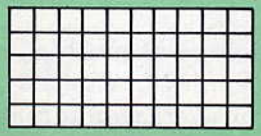
weeks and days tinted green	
____ wk., ____ da.	
twice as many	
____ wk., ____ da.	
1 wk., 3 da. more	half as many
____ wk., ____ da.	____ wk., ____ da.
× 3	÷ 3
____ wk., ____ da.	____ wk., ____ da.

F



time shown	
: PM	
2½ hours later	
:	
33 minutes later	10 hrs. later
:	:
1½ hrs. earlier	5 hrs. earlier
:	:

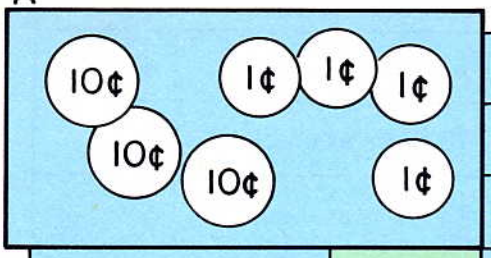
G



half as many	25	thirty-three less	
three times as many		twice as many	
sixty-seven more		× 4	

A

What are the coins worth?

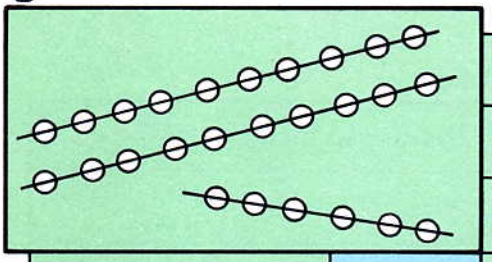


dimes shown	¢	three times as many dimes	¢	pennies shown × 7	¢
pennies shown	¢	three times as many pennies	¢	dimes shown × 7	$
amount shown	¢	three times amount shown	$	all coins × 7	$

pennies shown × 10	¢	pennies × 17	¢
dimes × 10	$	dimes × 17	$
all coins × 10	$	all coins × 17	$

B

How many beads?

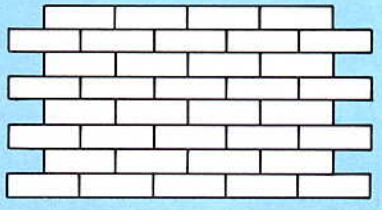


number shown		twice as many	
half as many		four times as many	
nine more		nine less	
3 times as many		5 times as many	
17 more		200 more	
1 less		40 more	

8 times as many	
217 more	
39 more	

C

How many bricks?

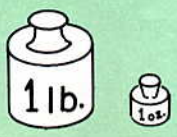


bricks shown		9 more	
7 more		79 more	
57 more		279 more	
twice as many		× 4	
ten times as many		× 5	
12 times as many		× 9	

÷ 2	
÷ 4	
÷ 6	

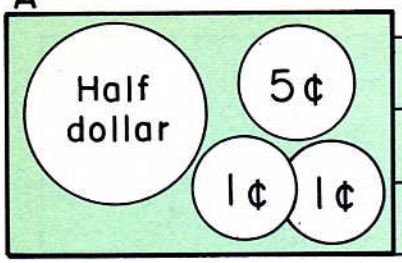
D

There are ____ ounces in one pound.



3 pounds	oz.	$1\frac{1}{2}$ pounds	oz.
10 pounds	oz.	$\frac{1}{4}$ pound	oz.
13 pounds	oz.	$1\frac{3}{4}$ pounds	oz.

A

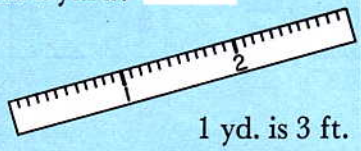


What are the coins worth?

35¢ more	¢	3 times as much	$
$8.35 more	$	10 times as much	$
$8.70 more	$	13 times as much	$

B

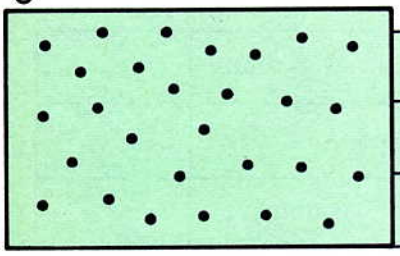
There are ____ inches in a yard.



1 yd. is 3 ft.

3 yards are	ft.	2 yards are	in.
½ yard is	in.	10 yards are	in.
7 yards are	ft.	12 yards are	in.

C



Number of dots

twice as many		10 times as many	
3 times as many		20 times as many	
5 times as many		30 times as many	

D

A row of B's

B B B B B B B B B B

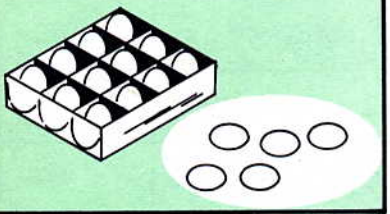
A row of M's

M M M M M M M M

How many letters

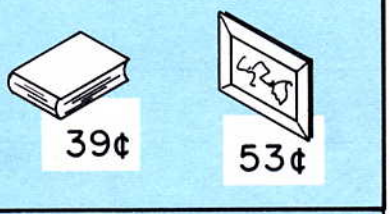
4 rows of B's		10 rows of each	
4 rows of M's		14 rows of each	
4 rows of each		7 rows of each	

E



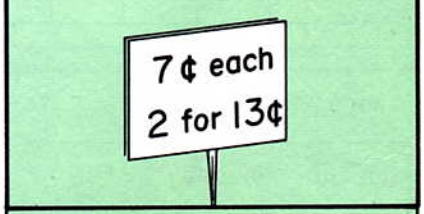
twice as many eggs
____ doz., ____ eggs
5 times as many
____ doz., ____ eggs
7 times as many
____ doz., ____ eggs

F



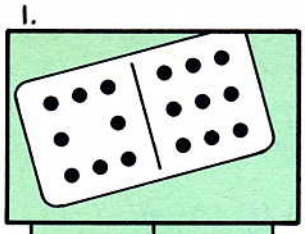
a book and a picture
¢
3 books and a picture
$
2 books and 3 pictures
$

G



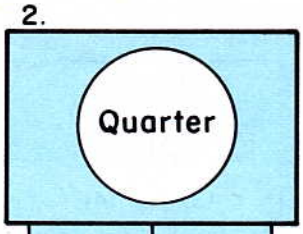
price of 5
¢
price of 7
¢
price of 12
¢

1.



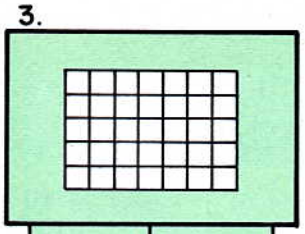
+ 8	− 9
25	
x 2	x 3
x 4	x 6

2.



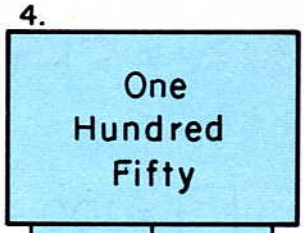
x 2	x 3
¢	¢
+ 18¢	− 6¢
¢	¢
+ 90¢	x 6
$	$

3.



+ 27	− 22
+ 127	x 2
x 10	x 12

4.



− 50	x 2
100	
+ 75	− 75
÷ 2	x 3

5.

Seven more than twenty − five	+ 8	x 2	+ 109
÷ 2	− 32	x 3	÷ 4
			14

6.

8	x 4	x 10	x 14	x 28
÷ 2	+	−	÷	x
	35	2	2	72

7.

	10	90	47		
+ 28	38			80	113

8.

	3	2	5	7	
x 9		18			108

9.

+ 6	x 2	÷ 2
30		
− 17	3 x 8	÷ 4
+	+	÷ 8
50	100	

10.

+	− 29	+ 372
50		
x 2	43	+ 20
		63
x 4	x 10	x 14

11.

+ 40	x 2	+ 79	÷ 2
100			30
÷ 1		÷ 4	÷ 5
÷ 3	− 27	÷ 6	÷ 10
	33		

"You didn't have many questions to ask. I tried to give you clues when I thought you might not understand my questions."

Ben had a comment. "That was a big mixture of things and numbers, and numbers without any particular things. But I knew all the time what you meant."

"That's fine," Mrs. Brown concluded. "We will come back to our Arithmetic of Things and Numbers when we talk about fractions."

$27 \div 3 = \square - 6$

(1) 20 (2) 15 (3) 24 (4) 9

Mark correct answer → (2)

Mark furthest from correct answer → (3)

(1) ○ (2) ● (3) ⊗ (4) ○ ____ ← Difference between answers you marked

1 $\square = 100 \div 4$

(1) 25 (2) 40 (3) 96 (4) 104

(1) (2) (3) (4) ○○○○ ____

2 $7 \times 6 = 8 + \triangle$

(1) 67 (2) 34 (3) 42 (4) 5

(1) (2) (3) (4) ○○○○ ____

3 $\bigcirc \times 5 = 4 \times 10$

(1) 20 (2) 19 (3) 40 (4) 8

(1) (2) (3) (4) ○○○○ ____

4 $32 \div 2 = \square \times \square$

(1) 8 (2) 16 (3) 4 (4) 30

(1) (2) (3) (4) ○○○○ ____

5 $13 + \triangle = 17 + 25$

(1) 42 (2) 29 (3) 65 (4) 0

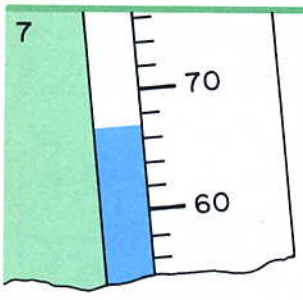
(1) (2) (3) (4) ○○○○ ____

6 $\square - 18 = \square \times 0$

(1) 0 (2) 18 (3) 36 (4) 81

(1) (2) (3) (4) ○○○○ ____

7



The temperature shown on the thermometer is

(1) 50° (3) $68\frac{1}{2}°$

(2) $63\frac{1}{2}°$ (4) 67°

(1) (2) (3) (4) ○○○○ ____°

8

If the temperature dropped 19°, it would be

(1) 38° (3) 48°

(2) 0° (4) 86°

(1) (2) (3) (4) ○○○○ ____°

9.

The time shown on the clock is

(1) 11:40 A.M. (3) 7:55 A.M.

(2) 11:38 A.M. (4) 8:05 A.M.

(1) (2) (3) (4) ○○○○ ____ hrs., ____ min.

10

How long is it until the clock will next read 12:00?

(1) 5 hrs., 50 min.

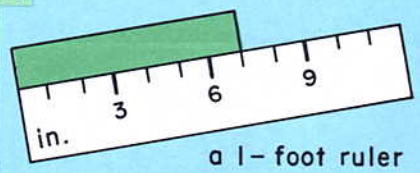
(2) 4 hrs., 5 min.

(3) 8 hrs., 55 min.

(4) 1 hr., 20 min.

(1) (2) (3) (4) ○○○○ ____ hrs., ____ min.

11



a 1-foot ruler

The length of a block 4 times as long would be

(1) 1 yd., 0 ft., 4 in.

(2) 0 yd., 2 ft., 4 in.

(3) 0 yd., 0 ft., 7 in.

(4) 0 yd., 1 ft., 9 in.

(1) (2) (3) (4) ○○○○ ____ ft., ____ in.

12

How many inches would have to be added to the green block in the sketch (in exercise 11) to make it 1 yard long?

(1) 29 (3) 23

(2) 36 (4) 31

(1) (2) (3) (4) ○○○○ ____

Mark correct answer

Mark furthest from correct answer

Difference between answers you marked

347 − 152

(1) 195 (2) 215 (3) 499 (4) 95

(1) ● (2) ○ (3) ⊗ (4) ○ _____

1

28 + 9 + 35 + 14

(1) 66
(2) 100
(3) 86
(4) 57

(1) (2) (3) (4) _____

2

324 − 78

(1) 402
(2) 246
(3) 354
(4) 242

(1) (2) (3) (4) _____

3

$3.57 × 2

(1) $12.40
(2) $3.59
(3) $1.57
(4) $7.14

(1) (2) (3) (4) $____

4

100 × 17 = _____

(1) 170 (3) 1700
(2) 107 (4) 710

(1) (2) (3) (4) _____

5

18 + 37 + 105 = ____

(1) 520 (3) 155
(2) 382 (4) 160

(1) (2) (3) (4) _____

6

125 ÷ 25 = _____

(1) 100 (3) 5
(2) 150 (4) 1

(1) (2) (3) (4) _____

Tickets: Weekdays — Adults..... $1.35 each, Children.....75¢ each
Weekends — Adults..... $1.65 each, Children.....85¢ each

7

Mrs. Wilson bought 3 adult tickets for Wednesday night's performance. She paid

(1) $3.35 (2) $2.60 (3) $3.95 (4) $4.05

(1) (2) (3) (4) $____

8

Mr. Henry bought 2 adult and 2 children's tickets for a weekend. He paid

(1) $5.70 (2) $5.00 (3) $7.50 (4) $6.35

(1) (2) (3) (4) $____

9

Mr. Jones decided to buy 7 adult tickets. How much would he save by buying them for a Tuesday rather than for a Saturday?

(1) $2.10 (2) 70¢ (3) $9.45 (4) $5.25

(1) (2) (3) (4) $____

10

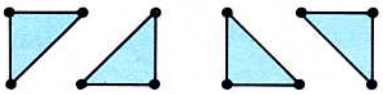
A class of 20 children bought tickets for a weekday. Because they went as a group, they paid 10¢ less per ticket than the regular price. How much did their 20 tickets cost?

(1) $12.75 (2) $8.50 (3) $13.00 (4) $15.00

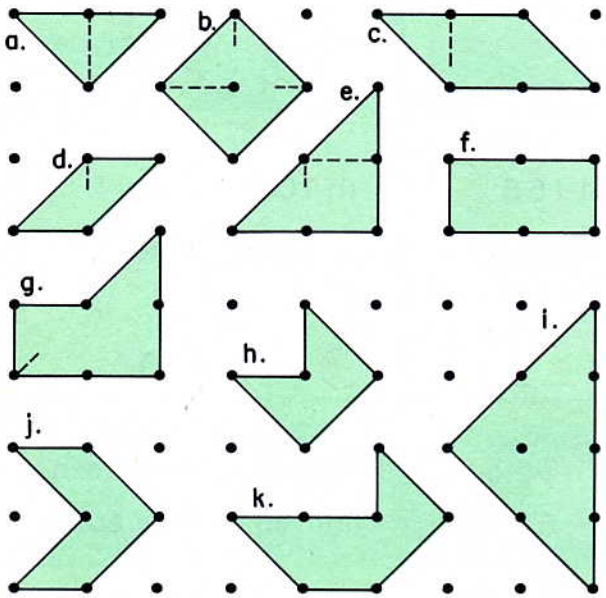
(1) (2) (3) (4) $____

1. How many pieces this size and shape

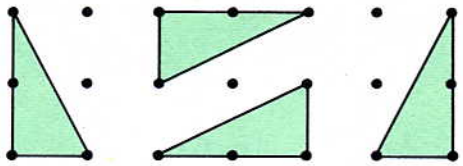
would it take to cover each of the shapes below? The pieces can be turned in any direction.



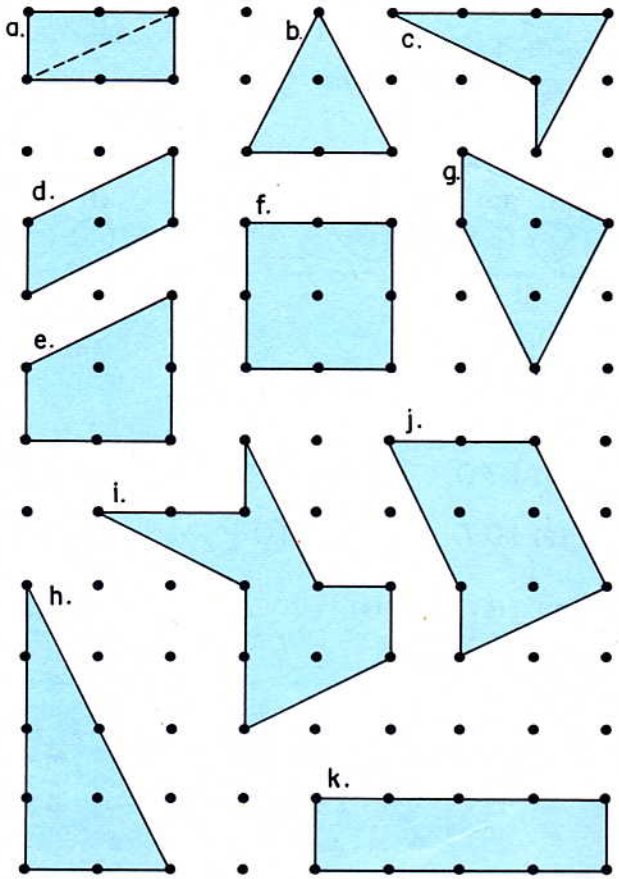
Use lines to show one of the ways you might do it.



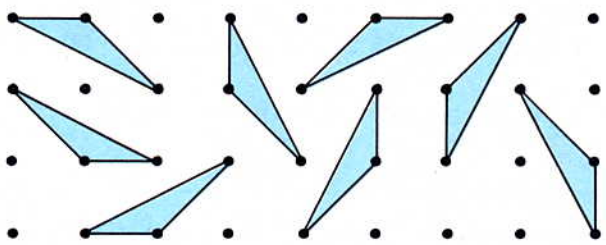
2. Show how many pieces this size and shape



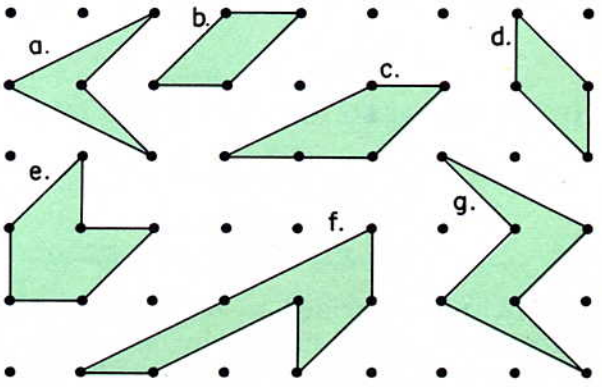
it would take to cover each of the shapes below.



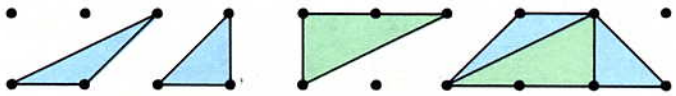
3. How many pieces this size and shape

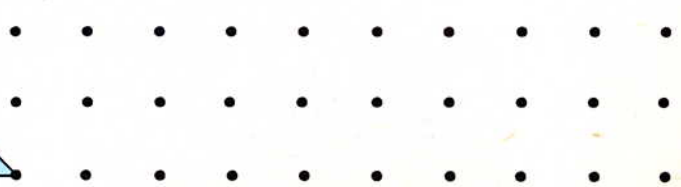


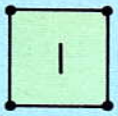
would it take to cover each of the shapes on the right?



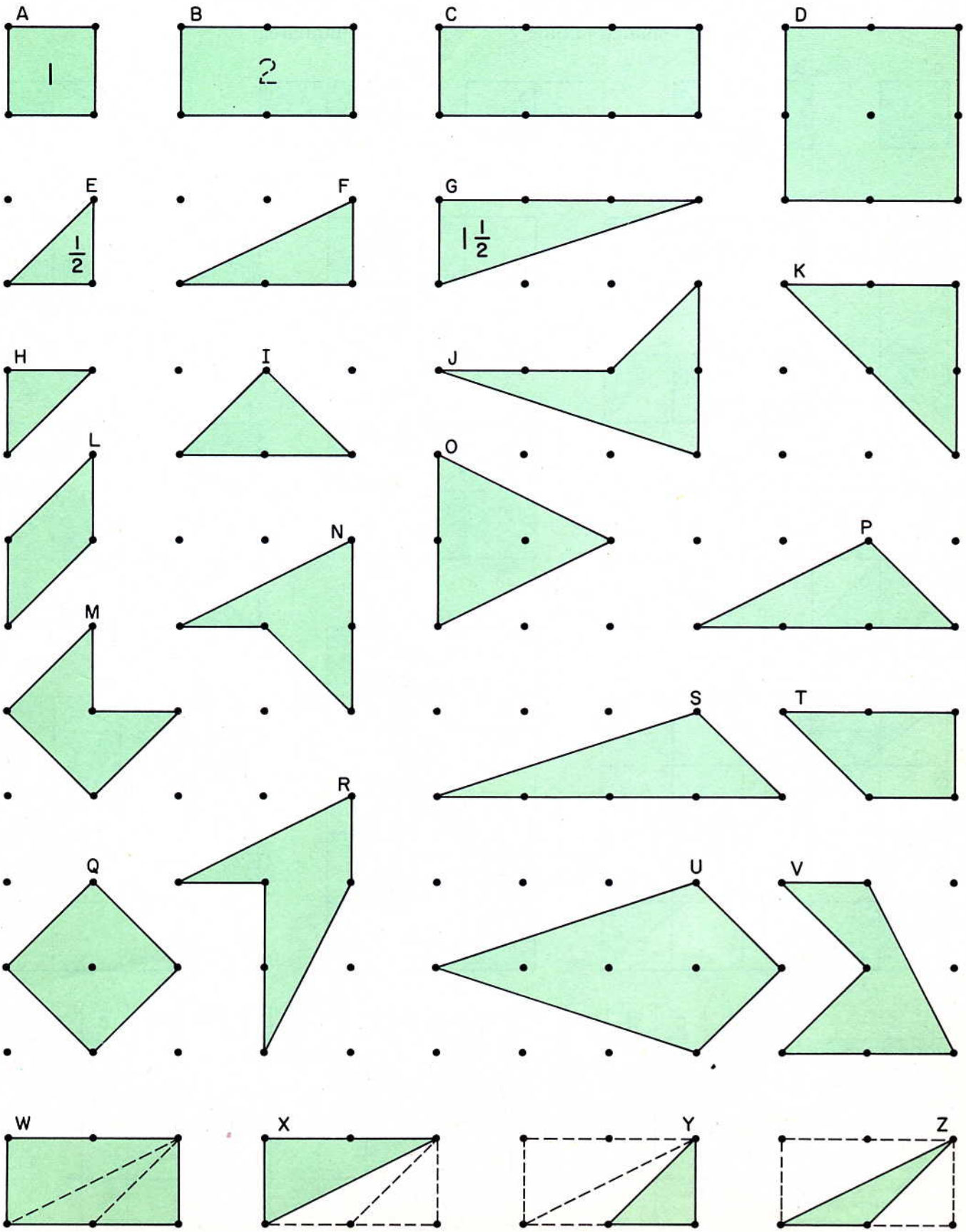
4. Make several designs with the three different triangles used above.

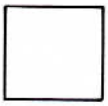




If we consider this green tinted space 

as one unit of area,
then how many units of area would each of the tinted spaces below represent?

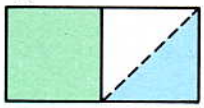


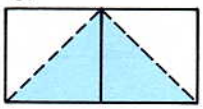
Let us agree that this  is 1 unit of area.

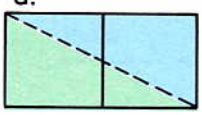
For each of the figures (a-q) below, tell the number of units of area of the

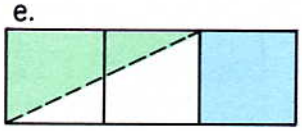
— area shaded in green
— area shaded in blue
— unshaded area
— total area

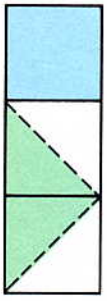
a.

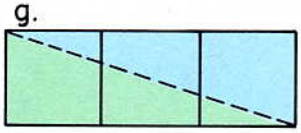
b. 

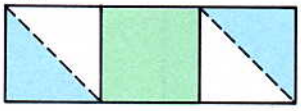
c. 

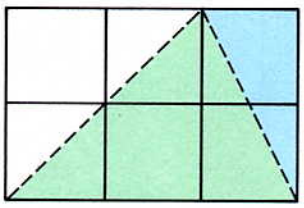
d. 

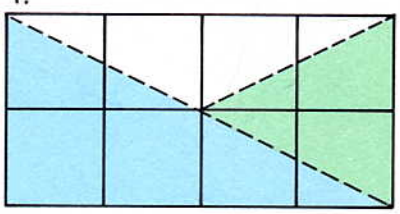
e. 

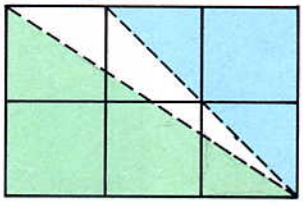
f 

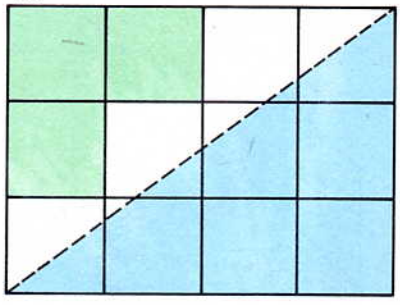
g. 

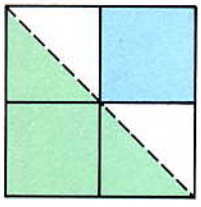
h. 

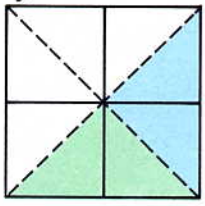
k. 

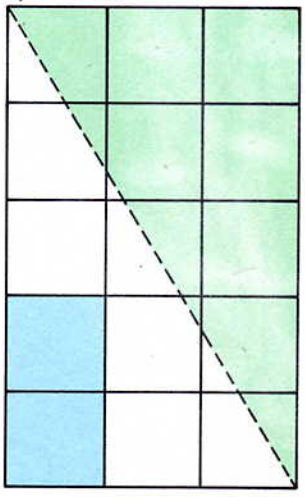
l. 

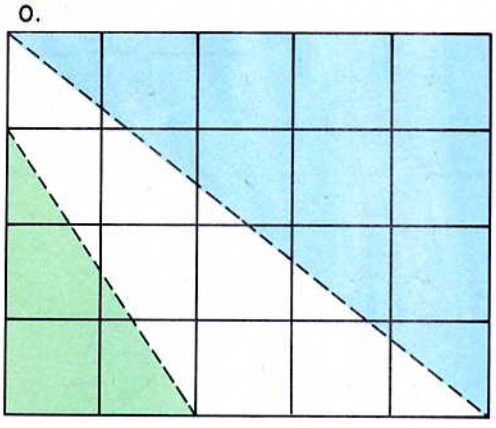
m. 

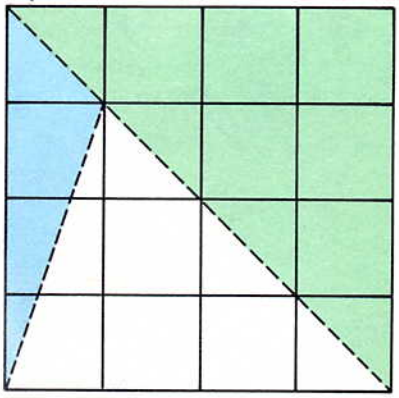
n. 

i. 

j. 

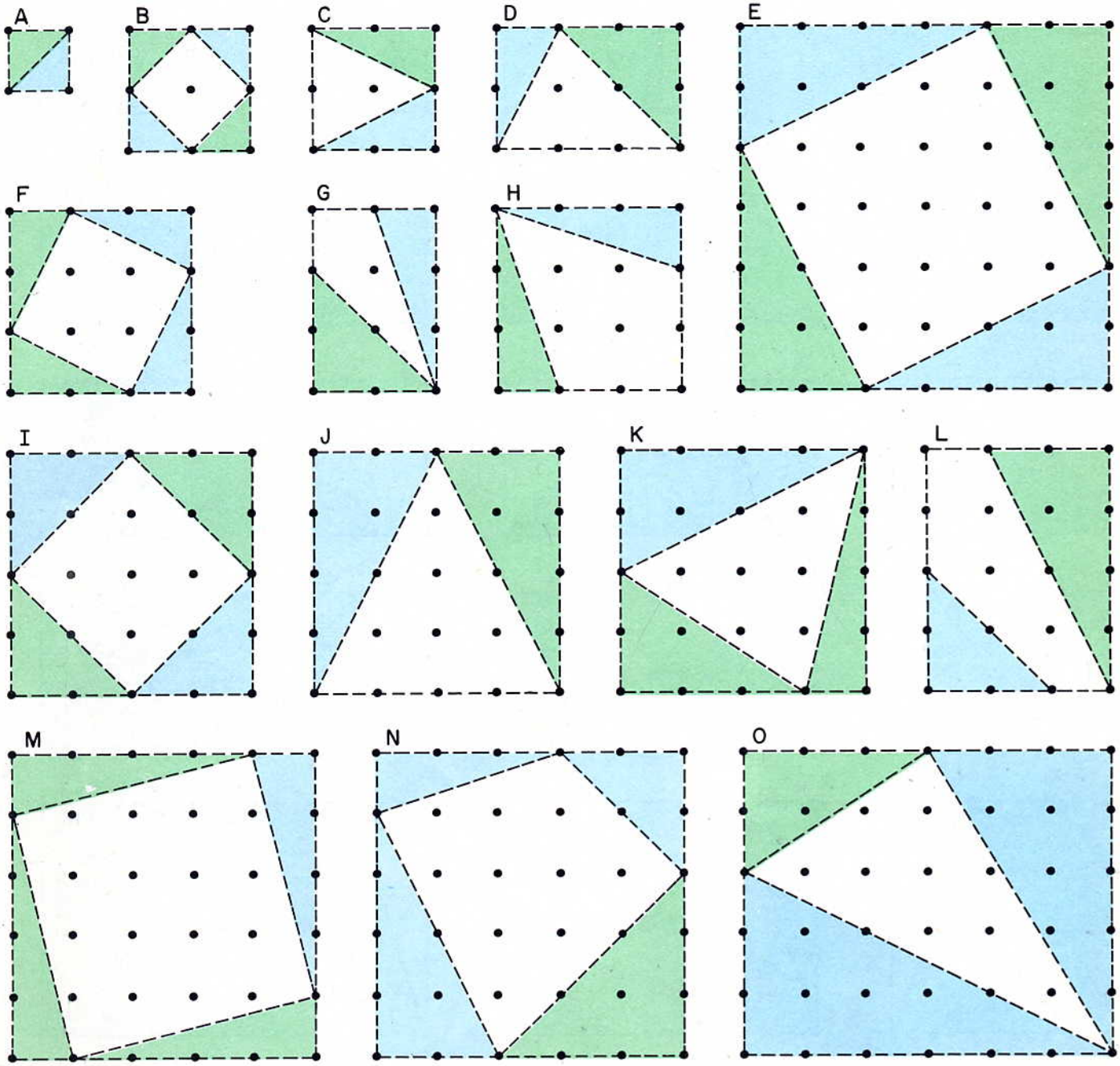
p. 

o. 

q. 

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
area shaded in green																	
area shaded in blue																	
unshaded area																	
total area																	

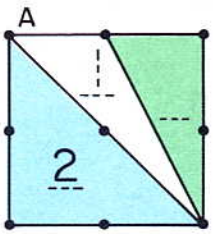
On this page we agree that is 1 unit of area.

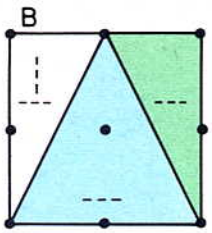


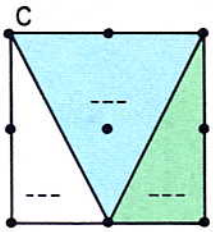
Units of area	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
shaded in green															
shaded in blue															
unshaded															
total															

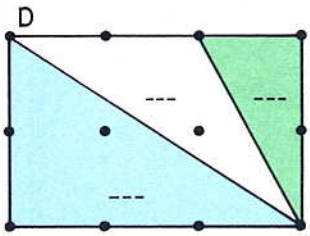
Indicate the size of each blue area first.

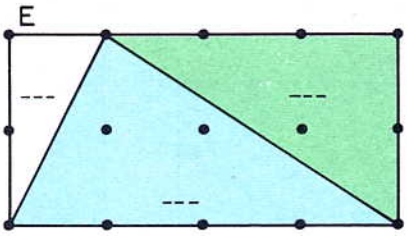
How many blocks this size would be needed to cover the spaces indicated below?

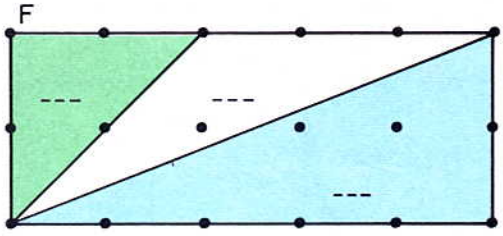


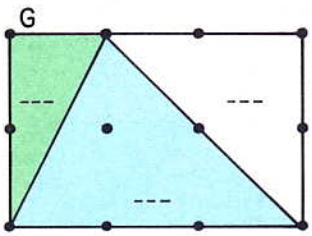


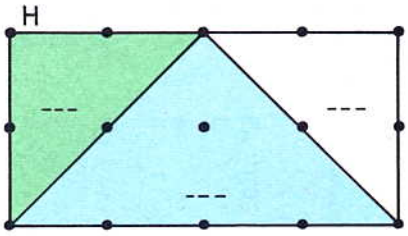


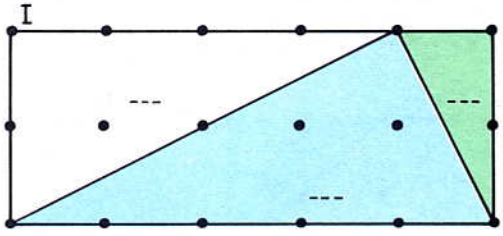


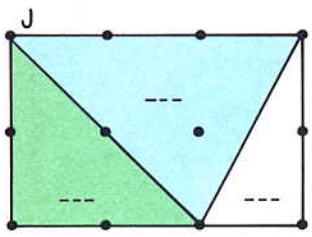


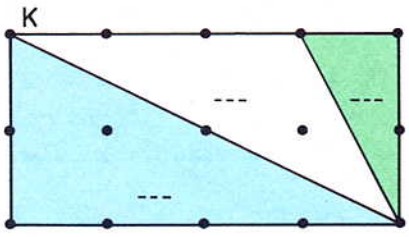


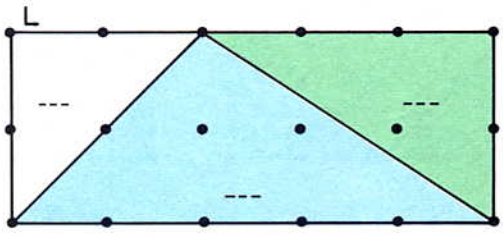


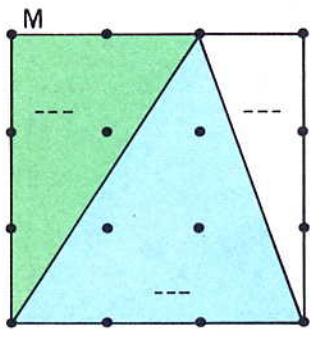


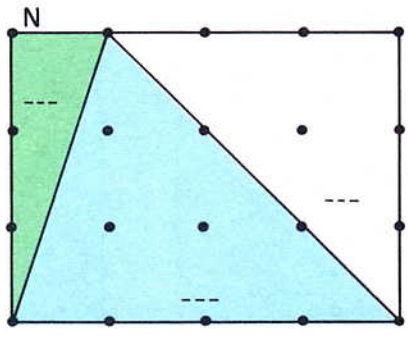


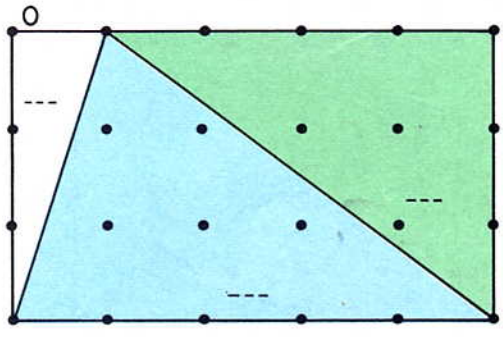




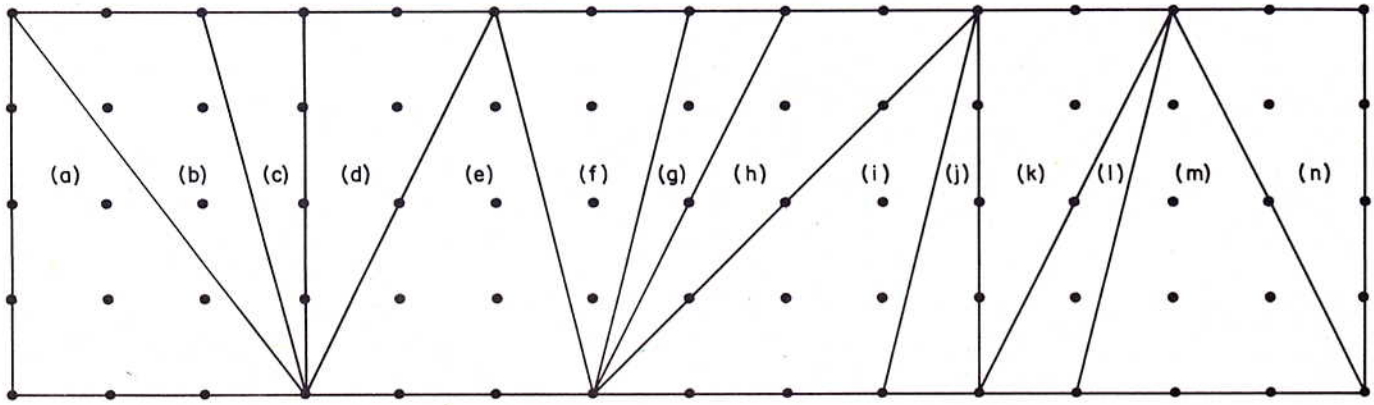






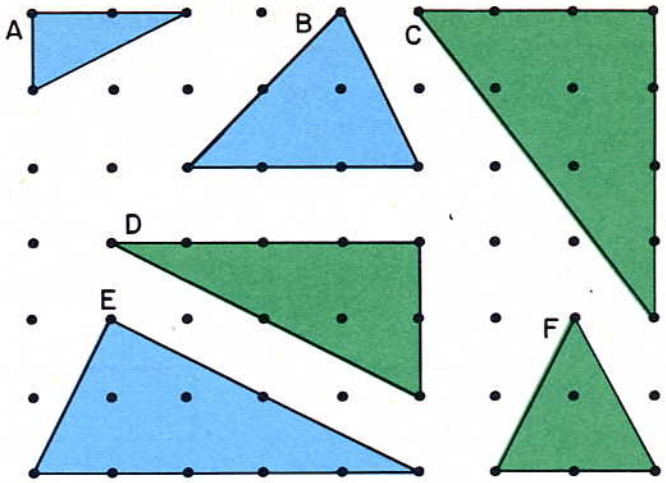


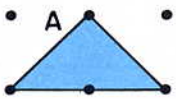
Color all shapes below that are the same size with the same color.

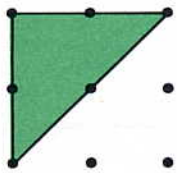


By connecting dots as in the examples given, draw at least three more triangles the same size as each example. Make all triangles different in shape. Two triangles are the same shape if one can be cut out and can be placed so that it exactly fits on the other. Otherwise they are of different shapes.

Label all triangles the same size with the letter shown on the right.

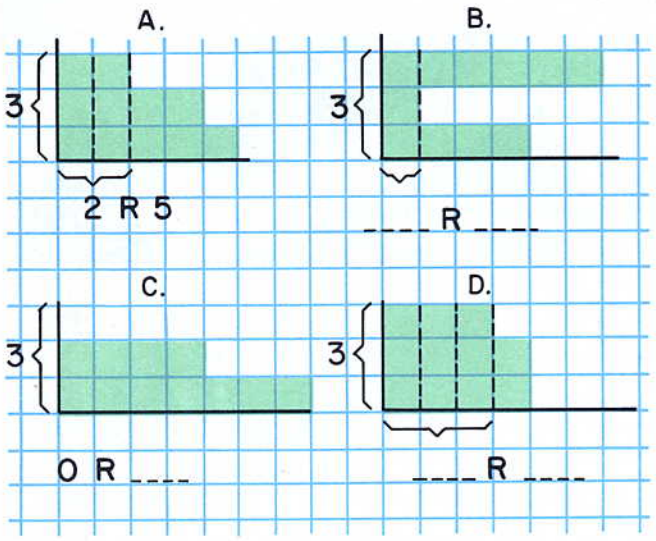






Arranging Tiles in Rows and Columns.

Here are several ways of arranging 11 tiles in 3 rows, and a shorthand method of recording these arrangements.



Arrangement D has more complete columns than the others. In other words, the "remainder" has the fewest tiles.

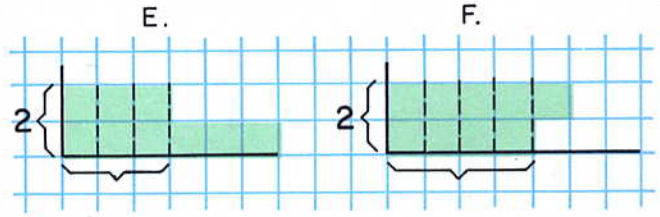
So, if we want the "most complete columns" or the "smallest remainder," only arrangement D would be a solution.

Here is a shorthand way to state a problem about arranging tiles:

$$2\overline{)9}$$

The shorthand states the question: What is the largest number of full columns (or the smallest remainder) if 9 tiles are arranged in 2 rows?

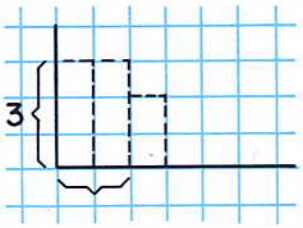
While both sketches below show 9 tiles arranged in 2 rows, only one has the most full columns (or the smallest remainder).



____ R ____ ____ R ____

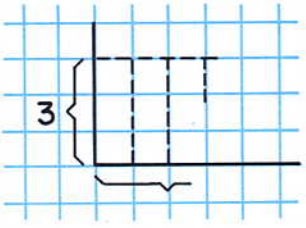
For each of the following examples, find the arrangement of tiles with the largest number of complete columns (or the smallest remainder).

G. $3\overline{)8}$



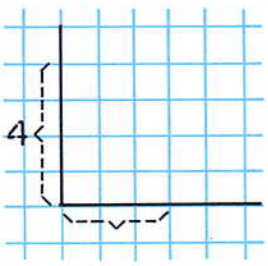
____ R ____

H. $3\overline{)14}$



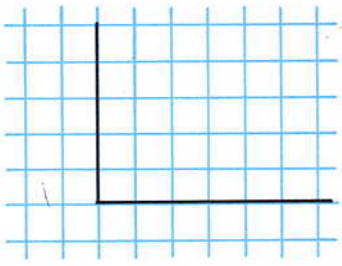
____ R ____

I. $4\overline{)12}$



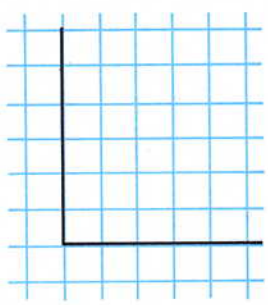
____ R 0

J. $4\overline{)18}$



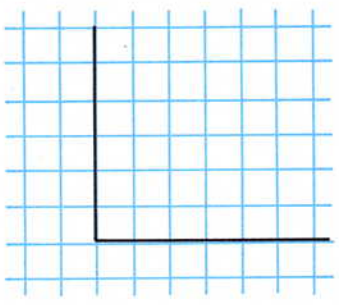
____ R ____

K. $5\overline{)17}$



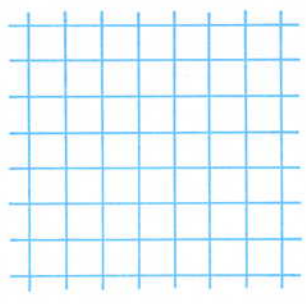
____ R ____

L. $5\overline{)24}$



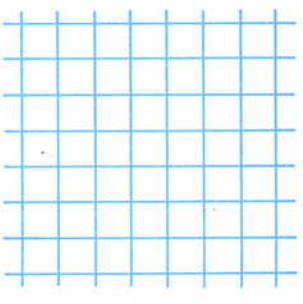
____ R ____

M. $3\overline{)16}$

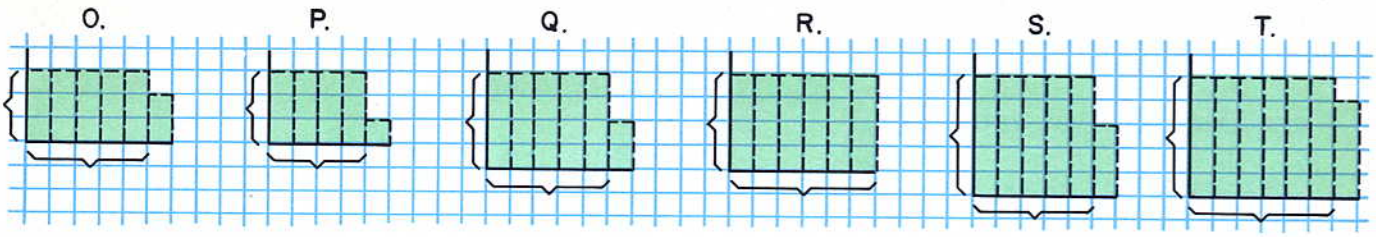


____ R ____

N. $4\overline{)14}$



____ R ____



Describe these arrangements in shorthand.

O.	P.	Q.	R.	S.	T.
3|17					
5 R	R	R	R	R	R

Make a sketch of each of the arrangements suggested below and complete the shorthand for each arrangement.

a.	b.	c.	d.	e.	f.	g.	h.
2|7	3|7	4|7	4|10	3|	5|14	|14	2|12
R	R	R	R	3 R 0	R	2 R 2	R

Complete the shorthand below.
Draw sketches only if you need to.

I	II	III	IV
3|25	4|25	5|25	6|25
R	R	R	R

V	VI	VII	VIII
|14	|20	4|30	9|30
3 R 2	5 R 0	R	R

IX	X	XI	XII
5|	6|	9|23	|32
3 R 2	5 R 0	R	6 R

XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX
|26	4|43	|39	8|	7|40	10|100	|50	1|
5 R	R	6 R 3	3 R 7	R	R	7 R 1	1 R 0

Another way of reporting arrangements of tiles in rows and columns.

Consider the following problems:

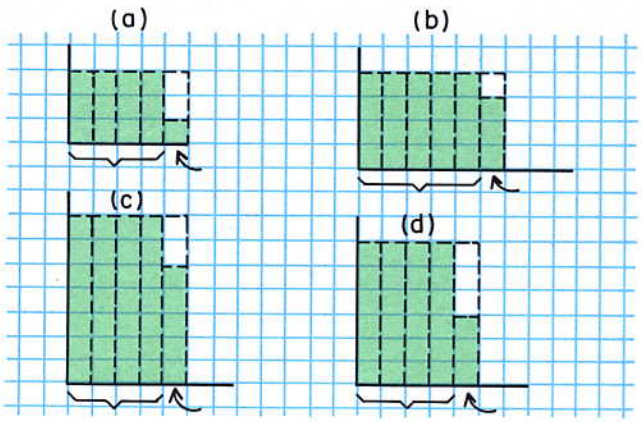
(a) arrange 13 tiles in 3 rows.

(b) arrange 23 tiles in 4 rows.

(c) arrange 33 tiles in 7 rows.

(d) arrange 27 tiles in 6 rows.

Sketches of the arrangements might look like this:



We can easily report the number of full columns and the remainders:

(a) ___R___ (b) ___R___

(c) ___R___ (d) ___R___ .

This method of reporting includes a record of the number of tiles in the incomplete column.

Another method of reporting tells us something else about the incomplete column. It tells us what part of the column is filled by the remaining tiles. Here is a way to describe this relation:

(a) $\frac{1}{3}$ (1 out of 3 or one-third full)

(b) $\frac{3}{4}$ (____ out of ____ or three-fourths full)

(c) —(____ out of ____ or five-sevenths full)

(d) —(____ out of ____ or _____ full)

These four problems with their results reported in this new way are usually written in this fashion:

(a)	(b)	(c)	(d)
$4\frac{1}{3}$	$5\frac{3}{4}$	$4\frac{-}{-}$	
$3\overline{)13}$	$4\overline{)23}$	$7\overline{)33}$	$6\overline{)27}$

This new way of reporting tells us something about arranging tiles in columns of a given size. But we would write the same reports if we were arranging marbles or cakes or pennies in groups of a given size. So we can read these reports in a general way:

(a) Thirteen divided by three equals four and one-third.

(b) Twenty-three divided by four equals five and three-fourths.

Look at example (d). We could have described the incomplete column (or group) in two ways, namely:

R3 (old way)

or:

$\frac{3}{6}$ or $\frac{1}{2}$ (new way)

since the incomplete column (or group)

has 3 tiles (or things) in it,

or

is three-sixths or one-half full.

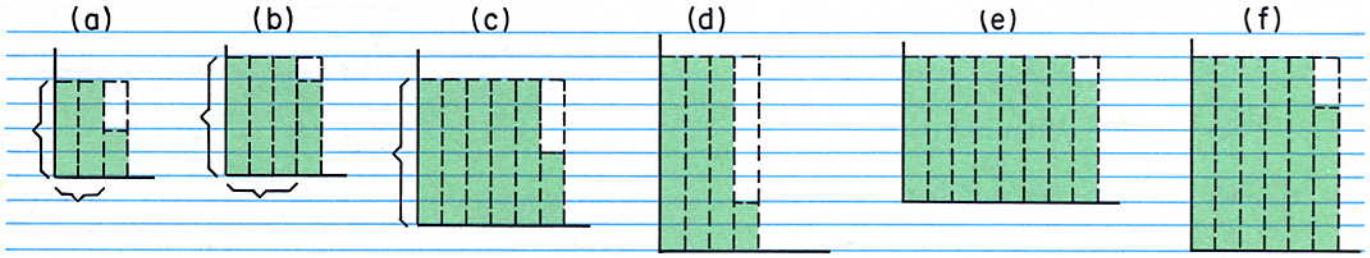
Please finish the following reports in the new shorthand. (In C and D report 14 divided by 4 in different ways.)

A	B	C	D
		$\frac{2}{-}$	
$3\overline{)17}$	$6\overline{)35}$	$4\overline{)14}$ or	$4\overline{)14}$

On this page use the more common way of recording results of arrangements of rows and columns and left over squares.

Former way: $3\overline{)14}$ 4R2 → More common way: $3\overline{)14}$ 4R2

Record each of the arrangements sketched below.
Show two or three different ways of recording each.



(a) $4\overline{)10}$ 2R, $4\overline{)10}$ $2\frac{2}{4}$, $4\overline{)10}$ $2\frac{1}{4}$

(b) $5\overline{)19}$ R, $5\overline{)\ \ }$

(c) $6\overline{)\ \ }$ R, $\overline{)\ \ }$, $\overline{)\ \ }$

(d) $\overline{)\ \ }$ R, $\overline{)\ \ }$, $\overline{)\ \ }$

(e) $\overline{)\ \ }$, $\overline{)\ \ }$

(f) $\overline{)\ \ }$, $\overline{)\ \ }$, $\overline{)\ \ }$

Use the grid lines above (if you need to) to draw sketches for any of the records g - o below.

(g) $4\overline{)30}$ R, $4\overline{)30}$, $4\overline{)30}$

(h) $5\overline{)44}$ R, $5\overline{)44}$

(i) $6\overline{)\ \ }$ 3R2, $6\overline{)\ \ }$, $6\overline{)\ \ }$

(j) $9\overline{)30}$ R, $9\overline{)30}$, $9\overline{)30}$

(k) $3\overline{)32}$, $\overline{)\ \ }$

(l) $\overline{)\ \ }$, $8\overline{)\ \ }$ $3\frac{2}{8}$, $\overline{)\ \ }$

(m) $10\overline{)45}$ R, $10\overline{)45}$, $10\overline{)45}$

(n) $4\overline{)49}$, $\overline{)\ \ }$

(o) $\overline{)\ \ }$, $\overline{)\ \ }$, $\overline{)21}$ $3\frac{1}{2}$

Following Directions

Some numbers are listed in the blue region. Make a list of pairs of these numbers which fit the following description.

14	3	5	56	23	18
19	12	17	66	2	
4	6	91	8	49	16

1. Their sum is 25 ..... 17, 8
2. Their product is 24 ..... 6, 4
3. Their difference is 13 ..... 17, 4
4. Their quotient is 4 ..... 8, 2
5. They are both odd numbers .....
6. They are both even numbers .....
7. Their sum is 10 and their product is 24 .....
8. Their sum is 7 and their difference is 1 .....
9. Their difference is 14 and their product is 51 .....
10. Their product is 36 and their quotient is 4 .....
11. Their difference is 47 and their sum is 85 .....

A nickel is worth

(1) 8¢ (2) 10¢ (3) 5¢ (4) 25¢

Mark correct answer

(1) (2) (3) (4)
○ ○ ● ○

12 Which number has a 7 in the tens place?

(1) 747 (3) 7,104
(2) 573 (4) 897

(1) (2) (3) (4)
○ ○ ○ ○

13 How long is it from 10:30 A.M. to 5:30 P.M.?

(1) 7 hrs. (3) 10 hrs.
(2) 5 hrs. (4) 15 hrs.

(1) (2) (3) (4)
○ ○ ○ ○

14 Which number is 4 ones, 6 tens, and 7 hundreds?

(1) 647 (3) 467
(2) 746 (4) 764

(1) (2) (3) (4)
○ ○ ○ ○

15 Which number comes next in this sequence? 23, 19, 15.

(1) 14 (3) 11
(2) 10 (4) 16

(1) (2) (3) (4)
○ ○ ○ ○

16 What is the cost of a dozen pens if 3 pens cost 35¢?

(1) 95¢ (3) $1.40
(2) 70¢ (4) $1.15

(1) (2) (3) (4)
○ ○ ○ ○

17 If 2 erasers cost 16¢, how many can you buy for 48¢?

(1) 6 (3) 8
(2) 3 (4) 24

(1) (2) (3) (4)
○ ○ ○ ○

18 17 x 3 =

(1) 51 (3) 37
(2) 73 (4) 42

(1) (2) (3) (4)
○ ○ ○ ○

19 2 is $\frac{1}{3}$ of what number?

(1) 23 (3) 7
(2) 222 (4) 6

(1) (2) (3) (4)
○ ○ ○ ○

20 What number is XXXVI?

(1) 51 (3) 53
(2) 36 (4) 34

(1) (2) (3) (4)
○ ○ ○ ○

In each of the Exercises I-IV, indicate the largest number by writing an A in the blue block, the next largest by a B, the next largest by a C, the smallest by a D, and then compute the differences asked for.

I

Seventy .....

Seven thousand .....

Seven hundred .....

Seven .....

difference between

A and B _____ B and D _____

B and C _____ C and D _____

A and C _____ A and D _____

II

Four hundred twenty-seven .....

Seven hundred forty-two .....

Two hundred seventy-four .....

Four hundred seventy-two .....

difference between

A and B _____ B and D _____

B and C _____ C and D _____

A and C _____ A and D _____

III

7×9 .....

$35 + 47$ .....

$100 \div 2$ .....

$207 - 109$ .....

difference between

A and B _____ B and D _____

B and C _____ C and D _____

A and C _____ A and D _____

IV

13 half-dollars .....

6 dollars and 17 dimes .....

4 dollars and 25 nickels .....

3 dollars and 18 quarters .....

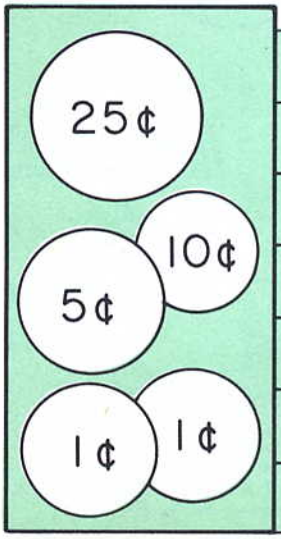
difference between

A and B $_____ B and D $_____

B and C $_____ C and D $_____

A and C $_____ A and D $_____

V



amount shown	¢	seventeen cents less	¢
twice as much	¢	thirty-nine cents more	¢
half as much	¢	needed to make 60¢	¢
three times as much	$	needed to make $2.00	$
one third as much	¢	eighty-five cents more	$
four times as much	$	number of coins	
ten times as much	$	10 times as many coins	

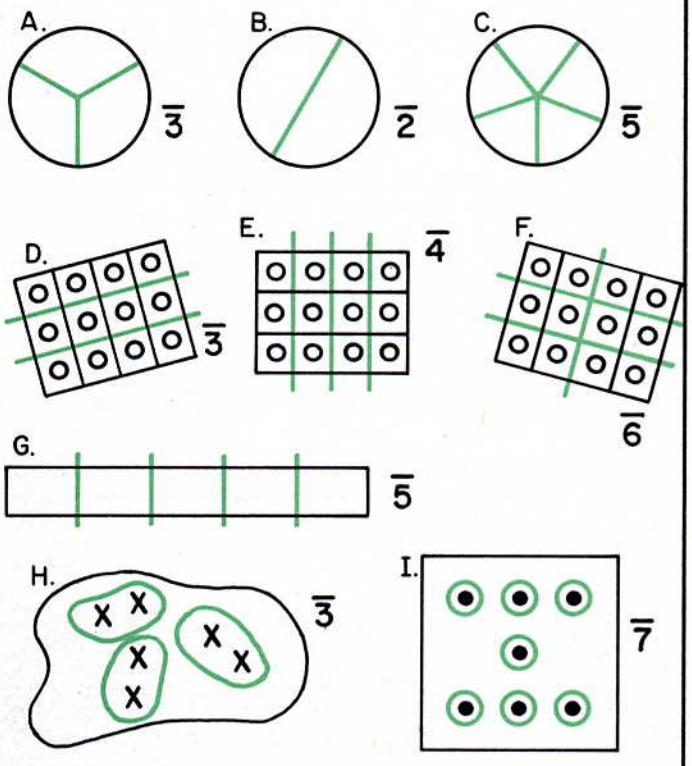
Ben asked Mrs. Brown whether she knew what he meant by this:

$\overline{3}, 2$

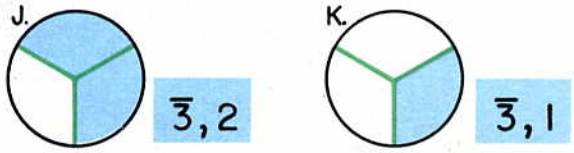
"No, I don't," Mrs. Brown was quick to admit. "What do you mean by that, Ben?"

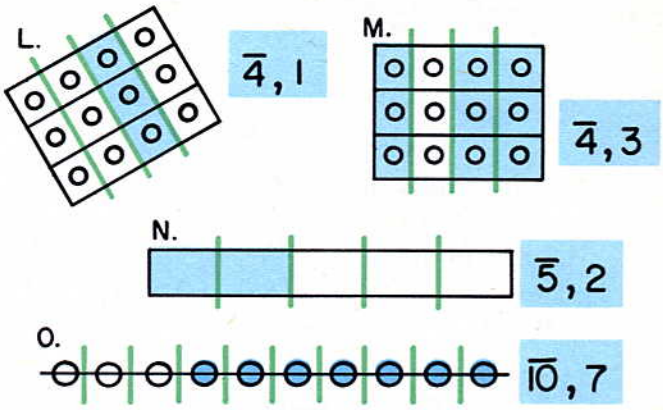
"I mean: cut (or separate) whatever we are talking about into 3 parts of equal size and color 2 of them — or paint out any 2 of them.

"First, I'll explain $\overline{3}$. I call that 'Bar three.' I call $\overline{2}$, 'Bar two'; $\overline{4}$, 'Bar four'; $\overline{7}$, 'Bar seven'; $\overline{12}$, 'Bar twelve'; etc.

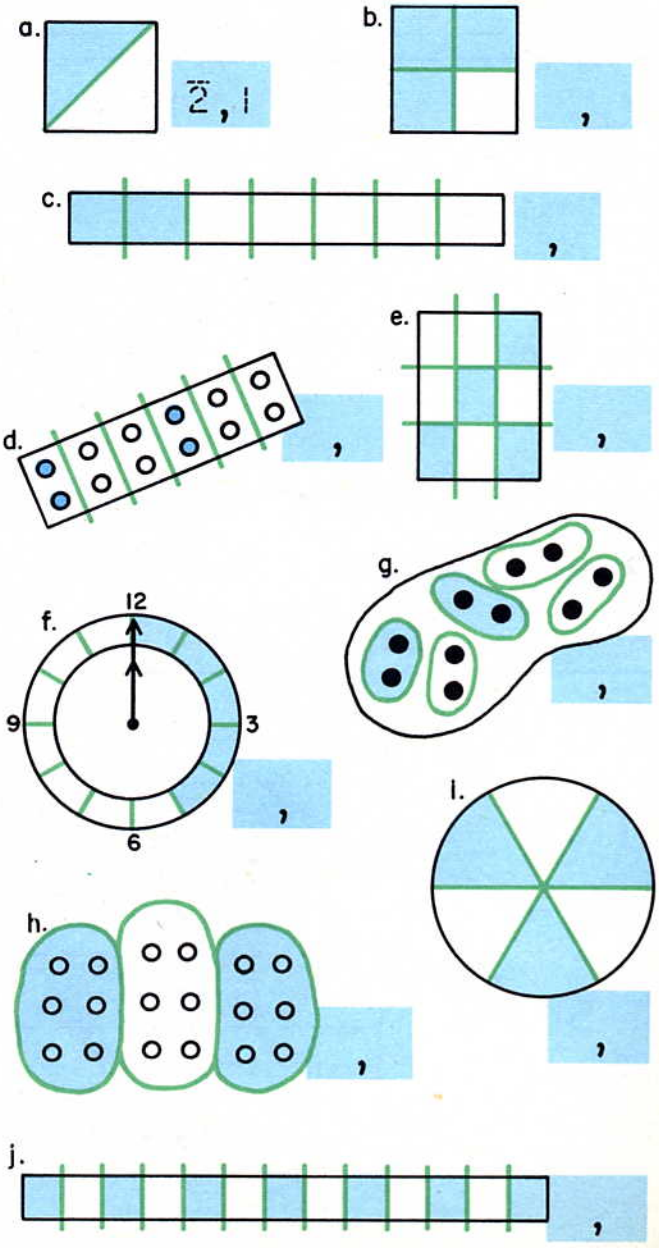


"The second number I write tells you to color a certain number of the parts I've cut or separated the original thing into.

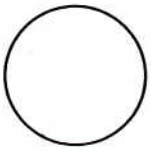




"Now I will make some sketches and you can write in the blue blocks the directions for cutting and coloring."



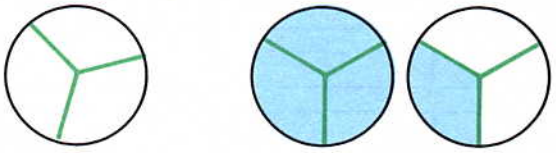
"It's very important," Ben told Mrs. Brown and the class, "to know what it is you're talking about. Suppose I gave you the following directions:



$\overline{3}, 4$

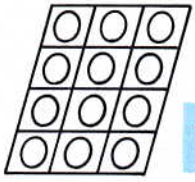
The $\overline{3}$ is easy enough. But how can you color in 4 parts if you only have 3 of them?

"Since I'm talking about a circle of the size shown, you will need to consider more than one of them.



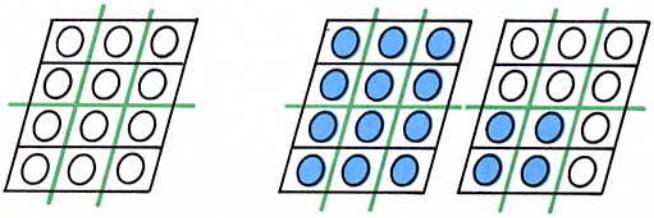
$\overline{3},$ and $\overline{3}, 4$

"Or suppose I was talking about a dozen eggs, and wrote:



$\overline{6}, 8$

"Again, I can do the $\overline{6}$, but I will need more dozens of eggs to show 8 such parts of a dozen."



$\overline{6},$ and $\overline{6}, 8$

At this point, Ruth interrupted.

"Really, Ben, I think you ought to tell everyone the simple way we write those directions."

"Don't tell us," Barbara said, "I think I've guessed."

"Let's hear your guess," Ben answered.

Barbara explained:

If you write $\overline{3}, 2$ .... I would write $\frac{2}{3}$

If you write $\overline{2}, 1$ .... I would write $\frac{1}{2}$

If you write $\overline{7}, 4$ .... I would write $\frac{4}{7}$

If you write $\overline{10}, 7$ .... I would write $\frac{7}{\quad}$

If you write $\overline{3}, 4$ .... I would write $\frac{\quad}{\quad}$

If you write $\overline{6}, 8$ .... I would write $\frac{\quad}{\quad}$

"Those are fractions," Harry exclaimed with surprise.

"Ruth always gives things away," Ben complained.

"Not really this time," Mrs. Brown suggested, "since Harry and some others in the class began to recognize your idea as a familiar one."

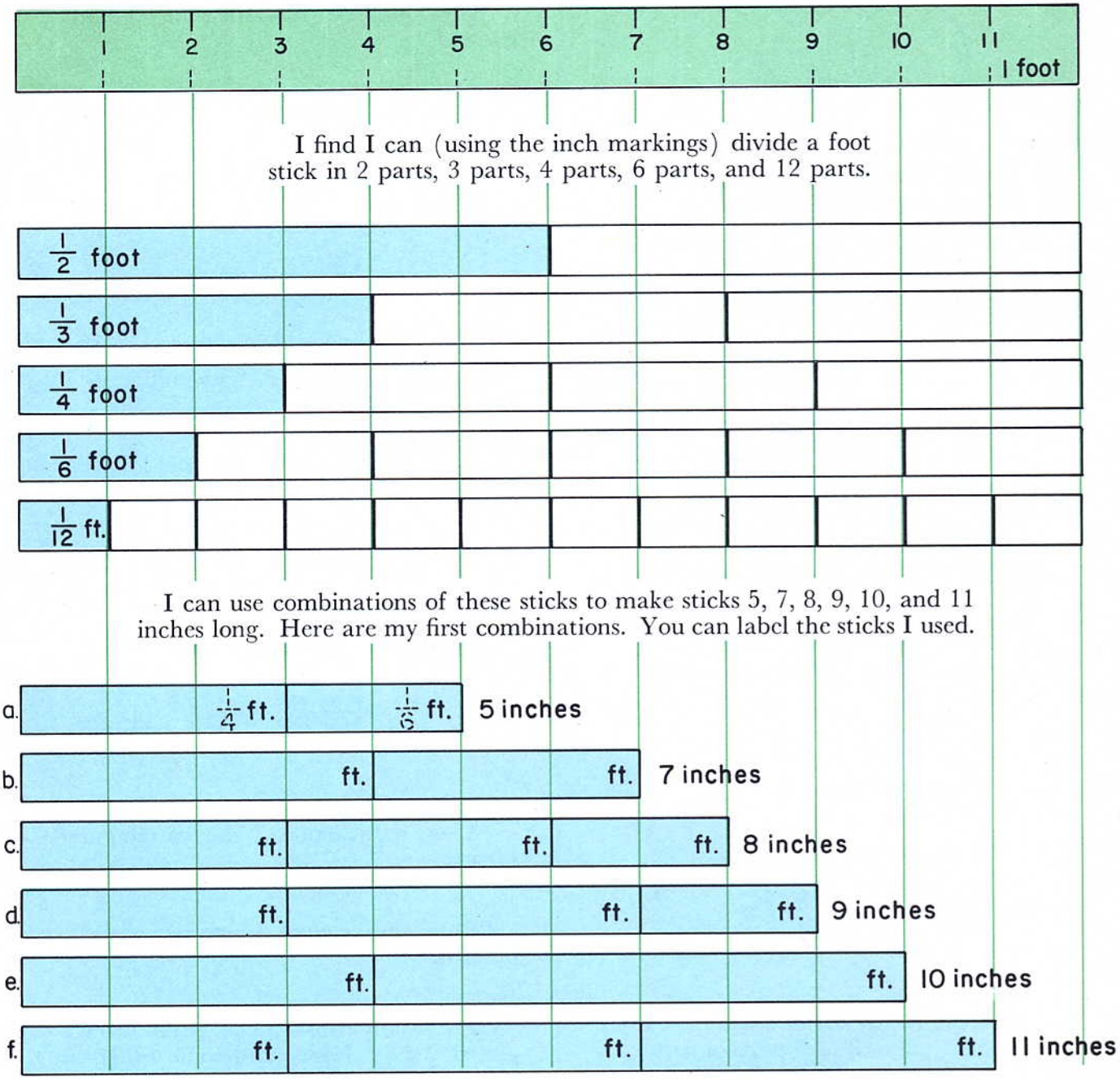
"Well," Ben said, "I've got some 'Iso Cards' I invented. They also have to do with the idea we are talking about. They are all about Things and Numbers."

Ben prepared the next several pages. Can you discover his rules?

Ben's Report

I am thinking about a 1 foot ruler.

Using the inch marks only, how many different ways can I divide the 1 foot ruler up into pieces of the same length? How can I talk about each length (in inches) as a part of a foot?


1 2 3 4 5 6 7 8 9 10 11
1 foot
I find I can (using the inch markings) divide a foot stick in 2 parts, 3 parts, 4 parts, 6 parts, and 12 parts.
1/2 foot
1/3 foot
1/4 foot
1/6 foot
1/12 ft.
I can use combinations of these sticks to make sticks 5, 7, 8, 9, 10, and 11 inches long. Here are my first combinations. You can label the sticks I used.
a. 1/4 ft. 1/6 ft. 5 inches
b. ft. ft. 7 inches
c. ft. ft. ft. 8 inches
d. ft. ft. ft. 9 inches
e. ft. ft. 10 inches
f. ft. ft. ft. 11 inches

Let's label these sticks to show the way they are made up:

A. 5 in.: $\frac{1}{4}$ ft. + $\frac{1}{6}$ ft.

B. 7 in.: — ft. + — ft.

C. 8 in.: — ft. + — ft. + — ft.

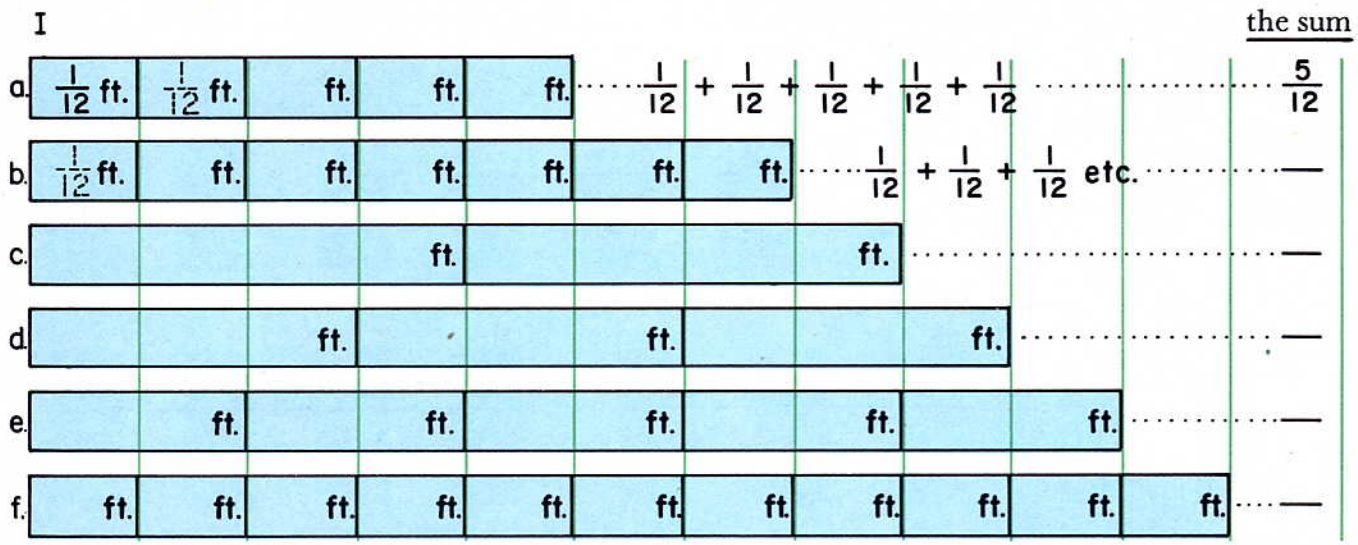
D. 9 in.: — ft. + — ft. + — ft.

E. 10 in.: — ft. + — ft.

F. 11 in.: — ft. + — ft. + — ft.

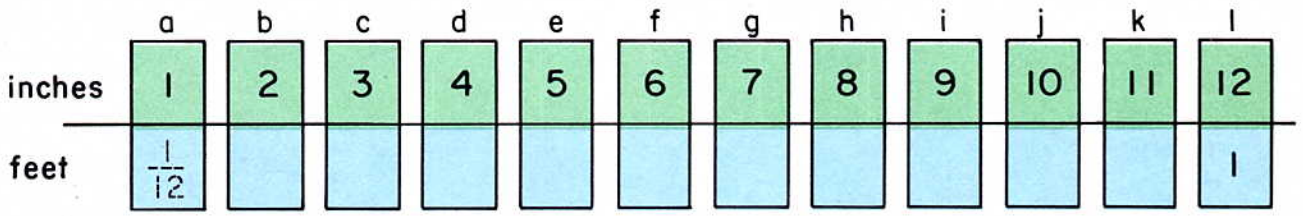
The labels I used were clumsy labels. So, I made each of the lengths (5, 7, 8, 9, 10, and 11 inches) out of the fewest pieces of equal length I could. I also made up new labels that told me which piece, and the number of such pieces I used.

I


the sum
a. 1/12 ft. 1/12 ft. ft. ft. ft. 1/12 + 1/12 + 1/12 + 1/12 + 1/12 5/12
b. 1/12 ft. ft. ft. ft. ft. ft. ft. 1/12 + 1/12 + 1/12 etc. —
c. ft. ft. —
d. ft. ft. ft. —
e. ft. ft. ft. ft. ft. —
f. ft. ft. ft. ft. ft. ft. ft. ft. ft. ft. ft. —

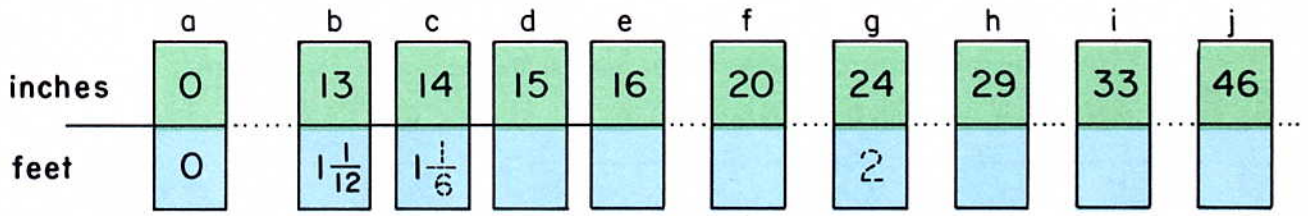
I made a set of what I call Iso Cards, listing a number of inches on one side, and wrote the new label on the other side. Please fill some of them in.

II



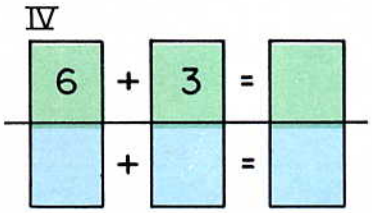
I made several of each card and expanded the deck by thinking about lengths that were longer than 12 inches. My plan for extending the deck will be obvious. Please help me.

III

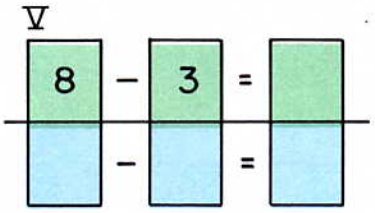


I use these cards to say things I know are true about inches. Please fill in the cards I've left blank.

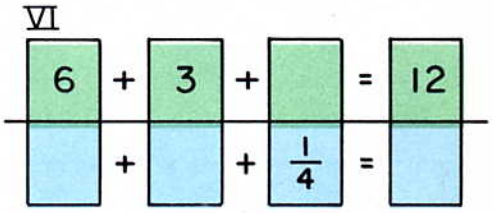
IV



V



VI



Now, it made sense for me to talk either about inches or about feet. I know that 6 inches plus 3 inches is 9 inches.

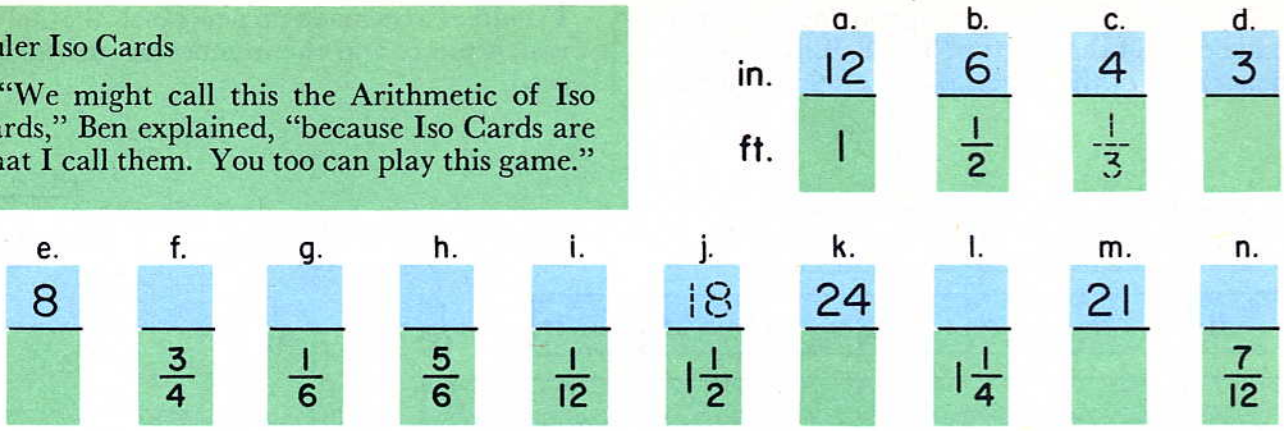
Then I can say:

$$\frac{1}{2}\text{ ft.} + \frac{1}{4}\text{ ft.} = \frac{3}{4}\text{ ft.}$$

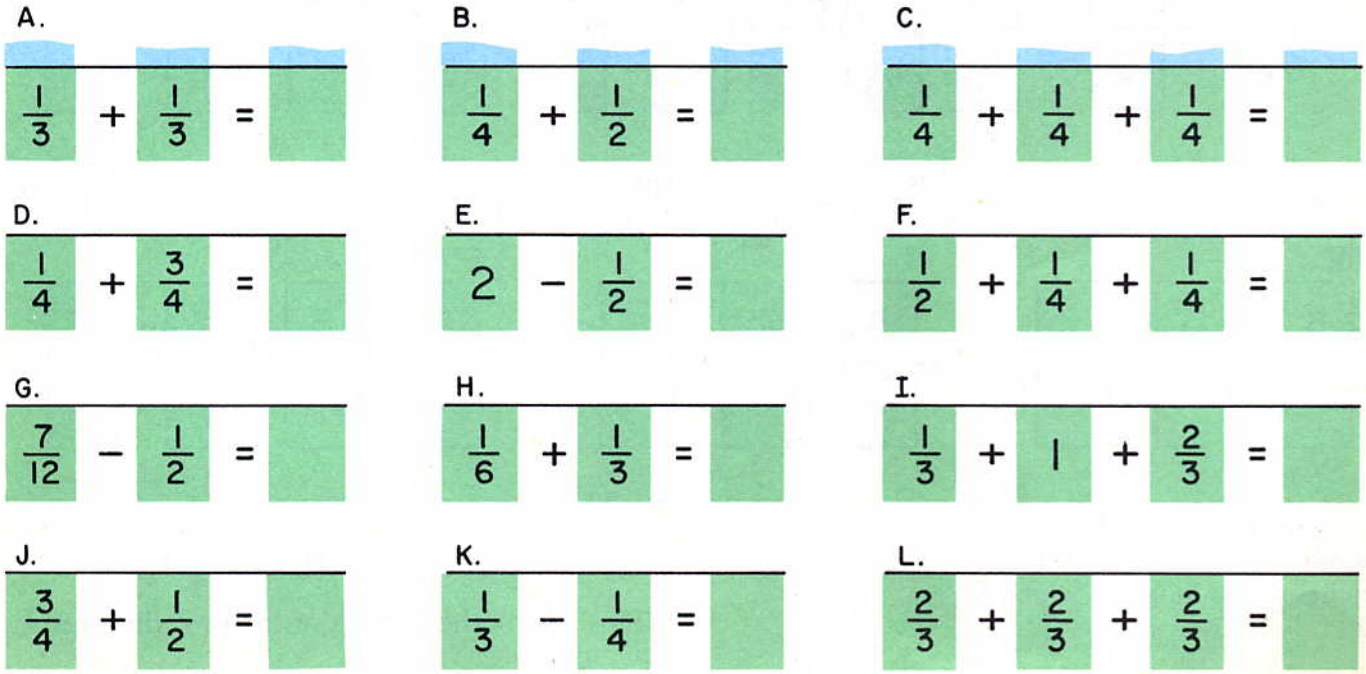
This is only the first set of cards of this type I made.

Ruler Iso Cards

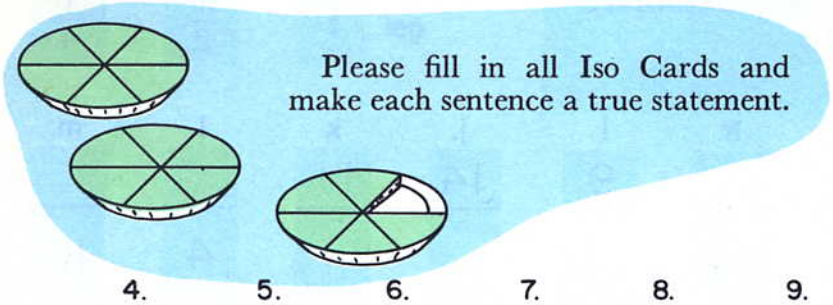
"We might call this the Arithmetic of Iso Cards," Ben explained, "because Iso Cards are what I call them. You too can play this game."



Below you see only the bottom portions of Iso Cards. Please fill in the green-tinted blocks so that each sentence is a true statement.



"I made other sets of cards," Ben explained, "when I thought of other things. Can you complete this new set of Iso Cards?"



Please fill in all Iso Cards and make each sentence a true statement.

	1.	2.	3.
pieces of pie	1	6	3
number of pies	$\frac{1}{6}$	1	$\frac{1}{2}$

	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.
	12	2	5	9	4		14		11	31
	2			$1\frac{1}{2}$		$1\frac{1}{6}$		$3\frac{2}{3}$		

14. $3 + 3 = \square$

$\square + \square = \square$

15. $4 + 2 = \square$

$\square + \frac{1}{3} = \square$

16. $1 + 2 + \square = \square$

$\square + \square + \frac{1}{2} = \square$

17. $6 - 2 = \square$

$\square - \square = \square$

18. $\square - \square = \square$

$1\frac{1}{2} - \frac{2}{3} = \square$

19. $\square + \square + \square = 12$

$\frac{5}{6} + \frac{1}{2} + \square = \square$

20. $\square + \square = \square$

$1\frac{1}{3} + 1\frac{5}{6} = \square$

21. $\square - \square = 8$

$\square - \frac{1}{3} = \square$

22. $\square + \square + 1 = \square$

$1\frac{1}{2} + 2\frac{2}{3} + \square = \square$

Below you see only the bottom portions of Iso Cards. Please fill in the green-tinted blocks so that each sentence is a true statement.

23. $\frac{1}{3} + \frac{1}{3} = \square$

24. $1 - \frac{1}{6} = \square$

25. $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \square$

26. $2\frac{1}{2} - \frac{2}{3} = \square$

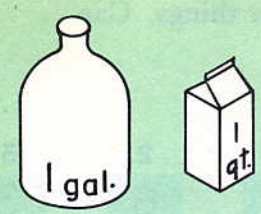
27. $\frac{5}{6} + 1\frac{5}{6} = \square$

28. $1\frac{1}{3} + \frac{5}{6} + 1\frac{1}{2} = \square$

29. $1\frac{1}{6} - \square = \frac{5}{6}$

30. $\square + \frac{1}{2} = \frac{5}{6}$

31. $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \square$



Use Ben's plan to fill in the tinted blocks so that each card is correct and each sentence is a true statement.

	a.	b.	c.
qt.	4	2	1
gal.	1	$\frac{1}{2}$	$\frac{1}{4}$

d.	e.	f.	g.	h.	i.
5		3			9
$1\frac{1}{4}$	$1\frac{1}{2}$		3	$1\frac{3}{4}$	

j.	k.	l.	m.
14	11		
		4	10

n.

1	+	1	=	2
$\frac{1}{4}$	+	$\frac{1}{4}$	=	$\frac{1}{2}$

o.

4	−		=	
	−		=	$\frac{3}{4}$

p.

3	+	3	+		=	
	+		+		=	$2\frac{1}{4}$

q.

	+	3	=	
$\frac{1}{4}$	+		=	1

r.

	−		=	1
	−	$\frac{1}{2}$	=	

s.

	+		+		=	6
	+		+	$\frac{3}{4}$	=	

t.

6	+		=	
	+		=	3

u.

	−		=	
2	−		=	$1\frac{1}{4}$

v.

	+		+	2	=	17
	+	2	+		=	

Below you see only the bottom portions of Iso Cards. Please fill in the green-tinted blocks so that each sentence is a true statement.

A.

$\frac{1}{2}$	+	$1\frac{1}{2}$	=	

B.

$\frac{1}{2}$	−	$\frac{1}{4}$	=	

C.

$1\frac{1}{2}$	+	$\frac{3}{4}$	+		= 3

D.

2	+	3	=	

E.

$\frac{3}{4}$	+	$\frac{3}{4}$	=	

F.

3	+	$2\frac{1}{2}$	+		= 7

G.

$1\frac{1}{4}$	+		=	$2\frac{1}{2}$

H.

$3\frac{1}{2}$	−	$\frac{3}{4}$	=	

I.

$\frac{1}{4}$	+		+	$1\frac{1}{4}$	= $1\frac{3}{4}$

J.

$4\frac{1}{2}$	+	$4\frac{1}{2}$	=	

K.

$10\frac{1}{4}$	+		=	$13\frac{1}{4}$

L.

$2\frac{1}{2}$	+	$2\frac{1}{2}$	+	$2\frac{1}{2}$	=	

Use Ben's plan to fill in the tinted blocks so that each card is correct and each sentence is a true statement.

cookies	18	9	6
boxes	1	$\frac{1}{2}$	$\frac{1}{3}$

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1	2	3	5	4		17			24
$\frac{1}{18}$	$\frac{1}{9}$				$\frac{2}{3}$		$\frac{5}{6}$	$\frac{4}{9}$	

11. $18 + 18 =$ ☐
☐ $+$ ☐ $=$ ☐

12. $2 + 6 =$ ☐
☐ $+$ ☐ $=$ ☐

13. ☐ $+$ ☐ $+$ ☐ $=$ ☐
$\frac{1}{9} + \frac{1}{9} + \frac{1}{9} =$ ☐

14. ☐ $+ 1 =$ ☐
$\frac{1}{18} + \frac{1}{18} =$ ☐

15. $7 + 8 =$ ☐
☐ $+$ ☐ $=$ ☐

16. ☐ $+$ ☐ $+$ ☐ $=$ ☐
$\frac{2}{9} + \frac{2}{9} + \frac{2}{9} =$ ☐

17. $20 -$ ☐ $=$ ☐
☐ $- \frac{4}{9} =$ ☐

18. ☐ $-$ ☐ $=$ ☐
$\frac{17}{18} - \frac{1}{2} =$ ☐

19. ☐ $+$ ☐ $+$ ☐ $=$ ☐
$\frac{1}{18} + \frac{1}{9} + \frac{1}{6} =$ ☐

Below you see only the bottom portions of Iso Cards. Please fill in the green-tinted blocks so that each sentence is a true statement.

20. $\frac{1}{2} + \frac{1}{3} =$ ☐

21. $\frac{1}{6} + \frac{1}{9} =$ ☐

22. $\frac{5}{18} + \frac{6}{18} + \frac{7}{18} =$ ☐

23. $\frac{2}{9} + \frac{1}{2} =$ ☐

24. $\frac{5}{9} - \frac{1}{6} =$ ☐

25. $\frac{1}{18} + \frac{1}{18} + \frac{1}{18} =$ ☐

26. $1\frac{1}{3} - \frac{1}{2} =$ ☐

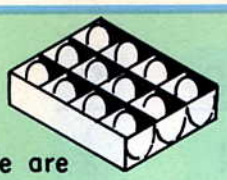
27. $1\frac{2}{9} -$ ☐ $= \frac{1}{2}$

28. $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} =$ ☐

29. $2\frac{1}{9} -$ ☐ $= 1\frac{5}{6}$

30. $3\frac{1}{3} - 1\frac{2}{3} =$ ☐

31. $1\frac{2}{3} + 1\frac{2}{3} + 1\frac{2}{3} =$ ☐



There are 12 eggs in 1 dozen.

	1.	2.	3.	4.	5.	6.	7.
eggs	6		4		3		10
doz.	$\frac{1}{2}$	1		$\frac{2}{3}$		$\frac{3}{4}$	

A.

$$\frac{1}{6} + \frac{1}{3} =$$

B.

$$\frac{7}{12} - \frac{1}{4} =$$

C.

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{2} =$$

D.

$$1\frac{1}{2} - \frac{2}{3} =$$

E.

$$\frac{1}{2} + \frac{5}{6} =$$

F.

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} =$$

G.

$$3 - 1\frac{5}{6} =$$

H.

$$2\frac{1}{3} - \frac{3}{4} =$$

I.

$$1\frac{1}{2} + 2\frac{1}{4} + 3\frac{1}{2} =$$

J.

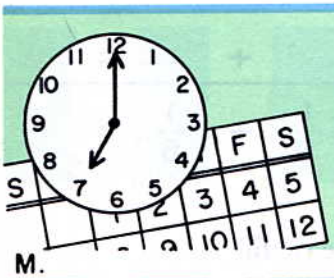
$$\frac{1}{12} + \frac{7}{12} =$$

K.

$$\frac{2}{3} + \frac{5}{12} =$$

L.

$$3\frac{1}{4} + \frac{2}{3} + 1\frac{2}{3} =$$



	I.	II.	III.	IV.	V.	VI.	VII.
hours	24		8		3		1
days	1	$\frac{1}{2}$		$\frac{1}{4}$		$\frac{1}{12}$	

M.

$$\frac{1}{6} + \frac{1}{3} =$$

N.

$$\frac{1}{8} + \frac{3}{8} =$$

O.

$$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} =$$

P.

$$\frac{1}{3} + \frac{1}{2} =$$

Q.

$$\frac{3}{8} - \frac{1}{3} =$$

R.

$$\frac{5}{24} + \frac{1}{24} + \frac{1}{24} =$$

S.

$$\frac{3}{4} + \frac{11}{12} =$$

T.

$$\frac{5}{8} + \frac{1}{4} =$$

U.

$$\frac{1}{8} + \frac{1}{4} + \frac{3}{8} =$$

V.

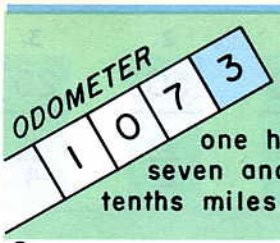
$$\frac{7}{24} + \frac{11}{24} =$$

W.

$$\frac{3}{8} + \frac{1}{6} =$$

X.

$$\frac{5}{6} + \frac{1}{3} + \frac{1}{2} =$$



one hundred seven and three tenths miles.

	1.	2.	3.	4.	5.	6.	7.
tenths	10	1	5	2	4	3	12
miles	1	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{5}$			

8. $1 + 1 = $; $\frac{1}{10} + $ $ = $

9. $2 + 4 = $; $\frac{1}{5} + $ $ = $

10. $ + $ $ + $ $ = $; $\frac{1}{5} + \frac{1}{5} + \frac{1}{10} = $

11. $ - $ $ = $; $\frac{9}{10} - \frac{7}{10} = $

12. $ + $ $ = $; $\frac{1}{2} + $ $ = 1\frac{1}{10}$

13. $ + $ $ + $ $ = $; $\frac{2}{5} + \frac{3}{5} + $ $ = 1\frac{7}{10}$

14. $\frac{7}{10} + $ $ = 1$

15. $2\frac{1}{2} - \frac{2}{5} = $

16. $1\frac{3}{5} + \frac{3}{10} + 1\frac{1}{2} = $

17. $\frac{2}{5} - \frac{3}{10} = $

18. $ + 1\frac{1}{10} = 2$

19. $7\frac{1}{10} + 10\frac{9}{10} + 4\frac{3}{5} = $

There are ☐ ounces in a pound.

	20.	21.	22.	23.	24.	25.	26.
ozs.	8	4	16		1		12
lbs.	$\frac{1}{2}$			$\frac{1}{8}$		$\frac{3}{8}$	

27. $ + 3 = $; $\frac{1}{16} + $ $ = $

28. $8 - $ $ = $; $ - $ $ = \frac{5}{16}$

29. $ + $ $ + $ $ = $; $\frac{1}{8} + \frac{1}{4} + $ $ = \frac{1}{2}$

30. $16 - $ $ = $; $ - \frac{7}{16} = $

31. $ + 8 = $; $\frac{1}{8} + $ $ = $

32. $ + $ $ + $ $ = $; $1\frac{1}{2} + \frac{1}{4} + \frac{3}{8} = $

33. $\frac{15}{16} + \frac{1}{16} = $

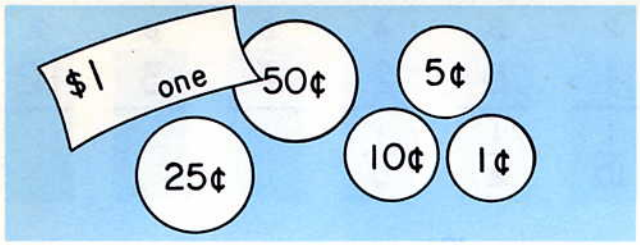
34. $ - \frac{1}{2} = \frac{3}{8}$

35. $1\frac{3}{16} + 2\frac{1}{4} + 1\frac{5}{8} = $

36. $\frac{1}{8} + $ $ = \frac{1}{4}$

37. $1\frac{3}{4} - $ $ = \frac{7}{8}$

38. $\frac{3}{8} + \frac{3}{16} + \frac{1}{2} = $



	1.	2.	3.
cents	100	50	25
dollars	1	$\frac{1}{2}$	$\frac{1}{4}$

4.	5.	6.	7.	8.	9.	10.	11.	12.	13.
	10		1	2	4	3	27	40	19
$\frac{3}{4}$		$\frac{1}{20}$							

14.

50	+	75	=				
$\frac{1}{2}$	+	$\frac{3}{4}$	=				

15.

17	+	33	=				
$\frac{17}{100}$	+	$\frac{33}{100}$	=				

16.

10	+	15	+		=			
	+	$\frac{3}{20}$	+	$\frac{1}{4}$	=	$\frac{1}{2}$		

17.

40	+		=				
	+		=	$\frac{13}{20}$			

18.

	+	35	=				
	+		=	$\frac{39}{100}$			

19.

	+	20	+		=			
$\frac{3}{10}$	+		+	$\frac{1}{4}$	=			

20.

150	+	210	=				
$1\frac{1}{2}$	+		=				

21.

	+		=				
$\frac{1}{4}$	+	$\frac{1}{5}$	=				

22.

	+		+		=			
$\frac{3}{10}$	+	$\frac{3}{10}$	+	$\frac{3}{10}$	=			

23.

$2\frac{1}{4}$	−	$1\frac{1}{2}$	=				

24.

$\frac{7}{10}$	−	$\frac{1}{5}$	=				

25.

$\frac{1}{20}$	+	$\frac{1}{10}$	+		=	$\frac{7}{20}$		

26.

$\frac{3}{10}$	+	$\frac{9}{10}$	=				

27.

$\frac{1}{2}$	+		=	$\frac{7}{10}$			

28.

$\frac{1}{4}$	+	$\frac{1}{2}$	+	$\frac{3}{4}$	=			

29.

$\frac{33}{100}$	+	$\frac{67}{100}$	=				

30.

$\frac{17}{100}$	−	$\frac{7}{100}$	=				

31.

$\frac{2}{5}$	+	$\frac{3}{5}$	+	$\frac{19}{100}$	=			

32.

$\frac{3}{25}$	+	$\frac{2}{25}$	=				

33.

$\frac{7}{20}$	−	$\frac{1}{10}$	=				

34.

$\frac{7}{100}$	+	$\frac{11}{100}$	+		=	$\frac{1}{4}$		

In 1 yard there are ☐ feet.

	1.	2.	3.	4.	5.	6.	7.
feet	3	6	1	2	9	4	5
yards	1		$\frac{1}{3}$				$1\frac{2}{3}$

8. $1 + 2 = $ ☐
$\frac{1}{3} + $ ☐ $ = $ ☐

9. ☐ $+ 2 = $ ☐
$\frac{2}{3} + $ ☐ $ = 1\frac{1}{3}$

10. $4 + $ ☐ $+ 4 = $ ☐
☐ $+ 1\frac{1}{3} + $ ☐ $ = 4$

11. $2 - $ ☐ $ = 1$
☐ $-$ ☐ $ = $ ☐

12. ☐ $- 1 = 6$
☐ $-$ ☐ $ = $ ☐

13. ☐ $+ 11 + $ ☐ $ = 20$
$2\frac{1}{3} + $ ☐ $ + \frac{2}{3} = $ ☐

14. $1\frac{1}{3} - \frac{2}{3} = $ ☐

15. $3\frac{1}{3} + 7\frac{2}{3} = $ ☐

16. $\frac{2}{3} + $ ☐ $ + \frac{2}{3} = 2$

17. ☐ $- 1\frac{1}{3} = 2\frac{2}{3}$

18. $4\frac{1}{3} - $ ☐ $ = 2\frac{2}{3}$

19. $1\frac{1}{3} + 2\frac{2}{3} + 7\frac{1}{3} = $ ☐

There are ☐ days in a week.

	20.	21.	22.	23.	24.	25.	26.
days	7	1	2	3	4	6	9
weeks	1	$\frac{1}{7}$	$\frac{2}{7}$				$1\frac{2}{7}$

27. $4 + 2 = $ ☐
$\frac{4}{7} + $ ☐ $ = $ ☐

28. ☐ $- $ ☐ $ = 3$
$1\frac{6}{7} - $ ☐ $ = $ ☐

29. ☐ $+$ ☐ $+$ ☐ $ = $ ☐
$\frac{1}{7} + \frac{2}{7} + \frac{3}{7} = $ ☐

30. $7 - 3 = $ ☐
☐ $-$ ☐ $ = $ ☐

31. ☐ $- $ ☐ $ = $ ☐
$2 - \frac{5}{7} = $ ☐

32. ☐ $+$ ☐ $+$ ☐ $ = $ ☐
$\frac{3}{7} + \frac{3}{7} + \frac{3}{7} = $ ☐

33. $\frac{6}{7} + 1\frac{1}{7} = $ ☐

34. $5 - \frac{2}{7} = $ ☐

35. $1\frac{2}{7} + 2\frac{6}{7} + $ ☐ $ = 5\frac{5}{7}$

36. $\frac{4}{7} + $ ☐ $ = 2$

37. $\frac{5}{7} + \frac{5}{7} = $ ☐

38. $\frac{5}{7} + \frac{5}{7} + \frac{5}{7} = $ ☐

Here are cards from three decks of Iso Cards.
Please fill in the incomplete cards below.

	1.	2.	3.	4.	5.	6.	7.	8.	9.
inches	6	12	3	4	1	8		9	
feet	$\frac{1}{2}$	1				$\frac{2}{3}$	$\frac{1}{6}$		$1\frac{1}{3}$

	10.	11.	12.	13.	14.	15.	16.	17.	18.
tenths of miles	5	10	3	4	1	2			20
miles	$\frac{1}{2}$	1	$\frac{3}{10}$	$\frac{2}{5}$			$\frac{4}{5}$	$1\frac{1}{2}$	

	19.	20.	21.	22.	23.	24.	25.	26.	27.
ounces	8	16	3	4	1	12			
pounds	$\frac{1}{2}$	1	$\frac{3}{16}$				$\frac{1}{8}$	$\frac{5}{8}$	$1\frac{3}{8}$

The three decks of Iso Cards shown above may give clues
that could help you to complete the sentences shown below.

28. $\square + \square = \square$

$\frac{1}{2} + \frac{1}{2} = \square$

29. $\square - \square = \square$

$\frac{1}{2} - \frac{1}{5} = \square$

30. $\square + \square + \square = \square$

$\frac{1}{6} + \frac{1}{3} + \frac{1}{4} = \square$

31. $\square + \square = \square$

$\frac{3}{8} + \frac{1}{4} = \square$

32. $\square + \square = \square$

$\frac{1}{2} + \frac{1}{3} = \square$

33. $\square + \square + \square = \square$

$1\frac{1}{2} + \frac{1}{5} + \frac{3}{10} = \square$

34. $\square - \square = \square$

$\frac{3}{4} - \frac{1}{8} = \square$

35. $\square - \square = \square$

$\frac{7}{10} - \frac{2}{5} = \square$

36. $\square + \square + \square = \square$

$1\frac{1}{3} + 2\frac{1}{6} + 4 = \square$

37. $\square + \square = \square$

$1\frac{1}{4} + 1\frac{3}{4} = \square$

38. $\square - \square = \square$

$1\frac{2}{5} - \frac{1}{2} = \square$

39. $\square + \square + \square = \square$

$10 + 3\frac{1}{2} + 1\frac{7}{8} = \square$

40. $\frac{7}{8} - \frac{3}{16} = \square$

41. $4 + \frac{5}{8} = \square$

42. $1\frac{2}{5} + 1\frac{7}{10} + 1\frac{1}{2} = \square$

Here are a few cards from several decks of Iso Cards.
Please fill in the incomplete cards in each deck.

	1.	2.	3.	4.	5.	6.	7.	8.	9.
eggs	6								
dozens of eggs	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{12}$	$1\frac{1}{3}$	$1\frac{5}{6}$

	10.	11.	12.	13.	14.	15.	16.	17.	18.
days	7	1	2	4	5	6	3	9	13
weeks	1	$\frac{1}{7}$							

	19.	20.	21.	22.	23.	24.	25.	26.	27.
cents	25	50							15
dollars	$\frac{1}{4}$		$\frac{3}{4}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{20}$	

The three decks of Iso Cards shown above may give clues
that could help you to complete the sentences shown below.

28. $\frac{4}{7} + \frac{2}{7} =$

29. $\frac{1}{2} + \frac{3}{4} =$

30. $\frac{1}{10} + \frac{3}{20} + \frac{1}{2} =$

31. $\frac{1}{3} - \frac{1}{4} =$

32. $1\frac{1}{4} - \frac{1}{20} =$

33. $\frac{3}{4} + \frac{3}{4} + \frac{3}{4} =$

34. $2 - \frac{3}{7} =$

35. $\frac{1}{5} + \frac{1}{4} =$

36. $\frac{5}{6} + \frac{5}{6} + \frac{5}{6} =$

37. $\frac{1}{2} + \frac{1}{6} =$

38. $\frac{3}{4} - \frac{2}{3} =$

39. $1\frac{4}{5} + 1\frac{4}{5} + 1\frac{4}{5} =$

40. $2\frac{7}{10} + \frac{2}{5} =$

41. $3\frac{2}{7} - 1\frac{4}{7} =$

42. $2\frac{3}{7} + 2\frac{3}{7} + 2\frac{3}{7} =$

43. $\frac{5}{6} - \frac{1}{3} =$

44. $\frac{1}{2} - \frac{1}{6} =$

45. $1\frac{4}{5} + 1\frac{3}{10} + 4\frac{1}{2} =$

I. Draw a loop around the pair that makes the sentence true.

Sample

□ x 6 = (4 x ○) + 2

□ 4, ○ 6 — □ 2, ○ 3

□ 5, ○ 7 — □ 0, ○ 1

A.

4 x 7 = (△ + □) − 9

△ 28, □ 10 — △ 15, □ 22

△ 19, □ 0 — △ 6, □ 6

B.

(4 x ○) + 4 = 5 x △

○ 9, △ 8 — ○ 16, △ 8

○ 5, △ 4 — ○ 7, △ 9

C.

△ x ○ = (5 x 9) − 3

△ 8, ○ 6 — △ 2, ○ 15

△ 6, ○ 7 — △ 13, ○ 3

D.

□ + □ = ○ x ○

□ 1, ○ 1 — □ 4, ○ 4

□ 0, ○ 7 — □ 2, ○ 2

E.

△ + □ = (7 x 8) + 4

△ 78, □ 4 — △ 23, □ 37

△ 26, □ 30 — △ 15, □ 4

F.

2 x (□ + △) = 5 x 6

□ 6, △ 9 — □ 10, △ 12

□ 2, △ 30 — □ 15, △ 7

G.

△ x ○ = 27 x 2

△ 13, ○ 0 — △ 9, ○ 6

△ 3, ○ 9 — △ 6, ○ 18

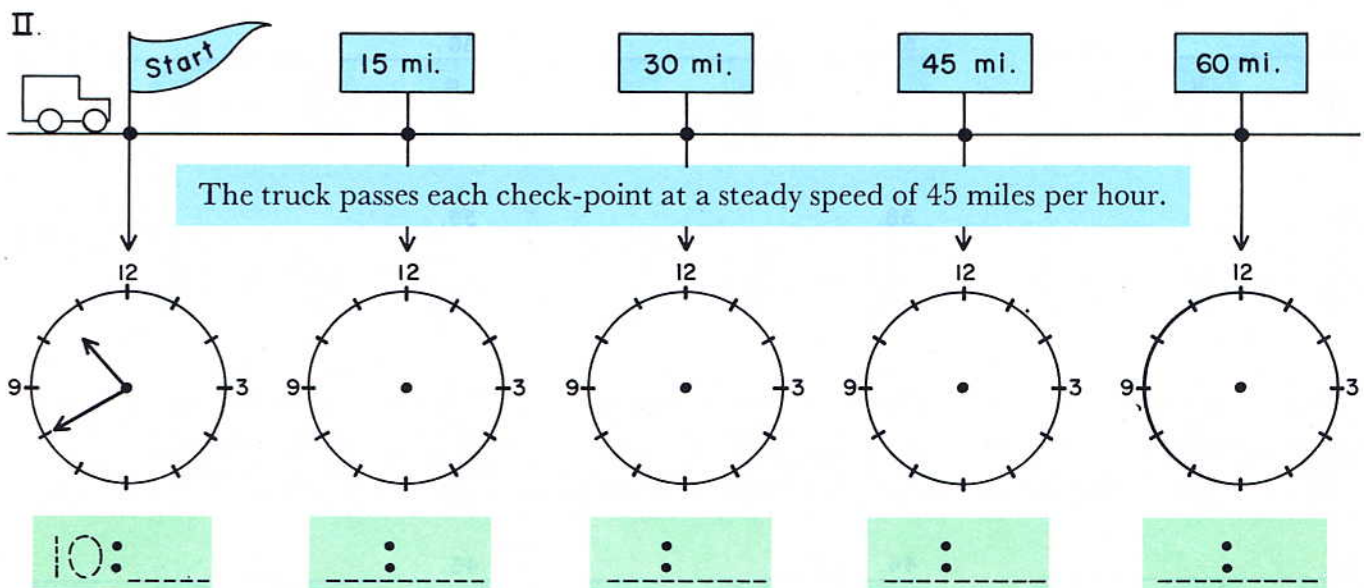
H.

(○ x 7) − 6 = (7 x □) − 13

○ 42, □ 91 — ○ 1, □ 6

○ 8, □ 9 — ○ 7, □ 10

II.



Show the hands of the clock and the time shown by the clock at each of the check-point marks.

Following Directions

$7 \times 7 =$ (1) 42 (2) 49 (3) 60 (4) 56

Mark the correct answer. Mark the furthest from the correct answer.

(1) (2) (3) (4) ○ ● ⊗ ○ ← The difference between the answers you marked.

1 A quarter is worth how much more than a dime?

(1) 35¢ (3) 15¢
(2) 41¢ (4) 10¢

(1) (2) (3) (4) ○ ○ ○ ○ _____¢

2 Bill earns 50¢ a week. How much does he earn in 13 weeks?

(1) $6.50 (3) $5.53
(2) $13.50 (4) $12.00

(1) (2) (3) (4) ○ ○ ○ ○ $_____

3 What is the cost of 2 dozen sheets of cardboard at 4 sheets for 25¢.

(1) $1.00 (3) $1.50
(2) $2.75 (4) $2.00

(1) (2) (3) (4) ○ ○ ○ ○ $_____

4 Cathy bought 2 cans of peas at 27¢ each and 3 cans of corn at 21¢ each. How much did she spend?

(1) $1.17 (3) $1.07
(2) 96¢ (4) $1.16

(1) (2) (3) (4) ○ ○ ○ ○ _____¢

5 How much change should Alice get back? She bought a book for 79¢ and a game for 90¢. She gave the clerk $2.00.

(1) $1.69 (3) 31¢
(2) $3.69 (4) 19¢

(1) (2) (3) (4) ○ ○ ○ ○ $_____

6 Harry bought 3 note books. "That's $\frac{1}{4}$ of all they had," he said. How many did the store have before Harry made his purchase?

(1) 7 (3) 4
(2) 16 (4) 12

(1) (2) (3) (4) ○ ○ ○ ○ _____

Sale

Belt	75¢	Scarf	$1.14
Blouse	$2.29	Dress	$3.58

7 A dress costs how much more than a blouse?

(1) $1.31 (3) $5.87
(2) $1.29 (4) 28¢

(1) (2) (3) (4) ○ ○ ○ ○ $_____

8 Four belts cost:

(1) $3.00 (3) $4.75
(2) $1.50 (4) $2.75

(1) (2) (3) (4) ○ ○ ○ ○ $_____

9 Altogether, one of each item would cost:

(1) $10.00 (3) $7.55
(2) $9.93 (4) $7.76

(1) (2) (3) (4) ○ ○ ○ ○ $_____

10 Two blouses and one scarf would cost:

(1) $4.54 (3) $5.72
(2) $3.42 (4) $4.73

(1) (2) (3) (4) ○ ○ ○ ○ $_____

11 Mary bought 2 blouses. Her change from a $10 bill should be:

(1) $7.71 (3) $5.42
(2) $3.12 (4) $3.13

(1) (2) (3) (4) ○ ○ ○ ○ $_____

12 Alice bought 2 blouses. Helen bought a scarf. How much more did Alice spend?

(1) $2.17 (3) $4.71
(2) $3.44 (4) $2.16

(1) (2) (3) (4) ○ ○ ○ ○ $_____

Jean Makes a Point

"I get a little tired." Jean objected to always having to think about Ben's things — his eggs and cookies and minutes and miles.

"I don't care what Ben is talking about . . .

> "I know that $\frac{1}{2}$ of anything plus $\frac{1}{4}$ of the same kind of thing equals $\frac{3}{4}$ of whatever you are talking about.

"So, I write:

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

and that is true unless someone wants to talk about special ideas . . . and if they do, they should explain.

"In each of the examples below, you can think of whatever things you like and the results will be the same."

1. $\frac{1}{5} + \frac{2}{5} =$ _____

2. $\frac{1}{3} + \frac{1}{6} =$ _____

3. $\frac{3}{4} + \frac{1}{4} =$ _____

4. $\frac{2}{3} + \frac{1}{4} =$ _____

5. $1\frac{1}{10} + 1\frac{2}{5} =$ _____

6. $\frac{5}{6} - \frac{1}{3} =$ _____

7. $2\frac{3}{4} + 1\frac{1}{2}$ _____

8. $3\frac{1}{3} - 1\frac{1}{2}$ _____

9. $7\frac{3}{10} + 2\frac{4}{5}$ _____

"Now," Jean continued, "I don't care what set of Ben's cards you think about . . . we all get the same results."

Mrs. Brown put each example on the board and asked for volunteers to give their answers.

"In the second example," Arleen said, "I wrote the answer differently. I was thinking of a pie that was cut into 6 pieces, so I wrote:

$$\frac{1}{3} + \frac{1}{6} = \frac{3}{6}$$

and I think that $\frac{1}{2}$ and $\frac{3}{6}$ mean the same thing. So, I think my answer is correct also."

Frank had a similar difference of opinion about example 5.

"I thought about miles — and tenths of a mile. I wrote:

$$1\frac{1}{10} + 1\frac{2}{5} = 2\frac{5}{10}$$

"I could have changed that to $2\frac{1}{2}$ miles, but people often talk about $2\frac{5}{10}$ miles . . . only they usually write 2.5 miles."

Everyone agreed that the way Arleen and Frank wrote their results made good sense.

Mary put the idea in words when she said

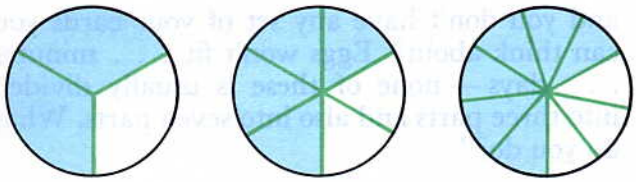
> "There are different ways to talk about the same thing."

And Ben, always ready to go back to his Arithmetic of Things and Numbers, asked to explain one of his ideas.

"Remember my

$\overline{3}$,2

way of giving directions? Well, I found out that many different directions would lead to the same general result. For example:



$\overline{3}$,2 $\overline{6}$,4 $\overline{9}$,6

"Each of these directions names the same amount of pie.

"Here is another example."

$\overline{2}$,1 $\overline{4}$,2 $\overline{6}$,3 $\overline{8}$,4

"Each direction refers to the same length of board."

Jean returned to the discussion. "Again, I think that Ben is too interested in special things — boards and pies. I think that I could say many things about:

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$$

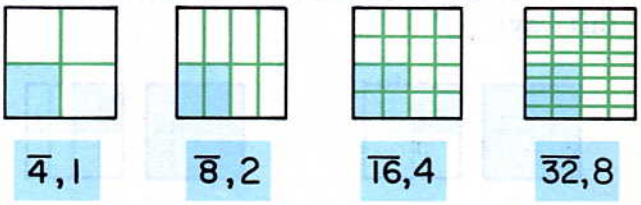
and:

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}$$

"As Mary said, they are different ways to talk about the same amount of whatever it is you are talking about."

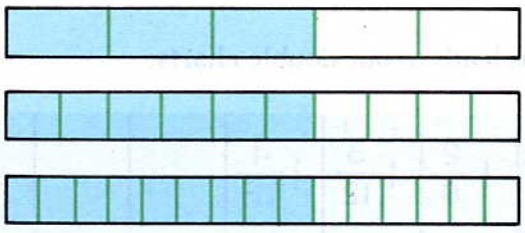
"Are there different ways to talk about fractions?" Al asked.

"I think so," was Ben's quick answer. "Because when you cut something up and talk about some of the pieces, you can always cut it up into smaller pieces and talk about these new pieces." He drew a sketch:



$\overline{4}$,1 $\overline{8}$,2 $\overline{16}$,4 $\overline{32}$,8

and



$\overline{5}$,3 —, —,

"Ruth and I used to make tables for Mother that looked like this. I'll leave blanks for you to complete.

A.

$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{\quad}$	$\frac{\quad}{8}$	$\frac{5}{\quad}$	$\frac{\quad}{12}$	$\frac{7}{\quad}$	$\frac{\quad}{16}$	$\frac{\quad}{\quad}$

B.

$\frac{2}{3}$	$\frac{4}{6}$	$\frac{6}{\quad}$	$\frac{\quad}{12}$	$\frac{10}{\quad}$	$\frac{\quad}{18}$	$\frac{14}{\quad}$	$\frac{\quad}{24}$	$\frac{\quad}{\quad}$

C.

$\frac{1}{4}$	$\frac{2}{8}$	$\frac{3}{\quad}$	$\frac{4}{\quad}$	$\frac{\quad}{20}$	$\frac{6}{\quad}$	$\frac{\quad}{28}$	$\frac{8}{\quad}$	$\frac{\quad}{\quad}$

D.

$\frac{3}{5}$	$\frac{6}{10}$	$\frac{9}{\quad}$	$\frac{12}{\quad}$	$\frac{\quad}{25}$	$\frac{18}{\quad}$	$\frac{\quad}{35}$	$\frac{24}{\quad}$	$\frac{\quad}{\quad}$

"And we always started our lists by noting the simplest way of writing about a fraction."

Ruth added to Ben's explanation.

"When fractions are greater than 1, we used to make a double list, because you can think of them in two different ways. Using our

$\overline{4}$,5

way of giving directions, we can sketch the idea this way:

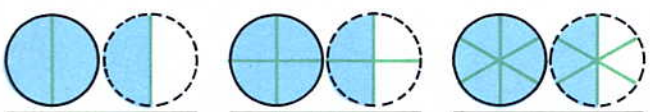
$\overline{4}$,5 $\overline{8}$,10

"This leads to our double charts:

A.

$1\frac{1}{4}$	$1\frac{2}{8}$	$1\frac{3}{12}$	$1\frac{4}{16}$		
$\frac{5}{4}$	$\frac{10}{8}$	$\frac{15}{12}$	$\frac{20}{}$		

"Here is another example."



$\overline{2}$,3 $\overline{4}$,6 $\overline{6}$,9

B.

$1\frac{1}{2}$	$1\frac{2}{}$	$\frac{}{6}$	$1\frac{}{8}$	$\frac{5}{}$	$\frac{6}{12}$
$\frac{3}{2}$	$\frac{6}{4}$	$\frac{9}{}$	$\frac{}{8}$	$\frac{15}{}$	$\frac{}{12}$

"Here is another without a sketch."

C.

$2\frac{1}{3}$	$2\frac{2}{}$	$2\frac{}{9}$	$\frac{4}{}$	$\frac{}{15}$	
$\frac{7}{3}$	$\frac{14}{}$	$\frac{}{9}$	$\frac{28}{}$	$\frac{}{15}$	

Jack had a question for Ben.

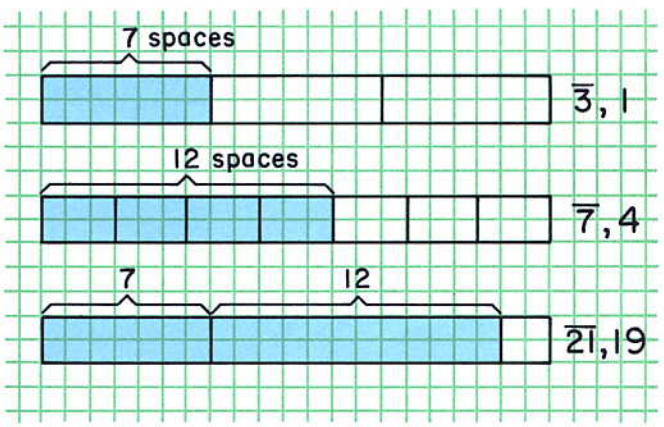
"What do you do, Ben, when you run across a problem such as

$$\frac{1}{3} + \frac{4}{7}$$

and you don't have any set of your cards you can think about? Eggs won't fit . . . minutes . . . days — none of these is usually divided into three parts and also into seven parts. What do you do?"

"Well, I used to work out something using a piece of graph paper. I wanted to draw a picture of a stick that I could easily cut into 3rds and into 7ths. How long should I draw it?

"What I really wanted to do was find a number that could be easily divided by 3 and by 7. Of course, 21 would do. (So would 42.) So I drew my stick 21 spaces long. And you can see my method:



"So, I would conclude that

$$\frac{1}{3} + \frac{4}{7} = \frac{19}{21}.$$

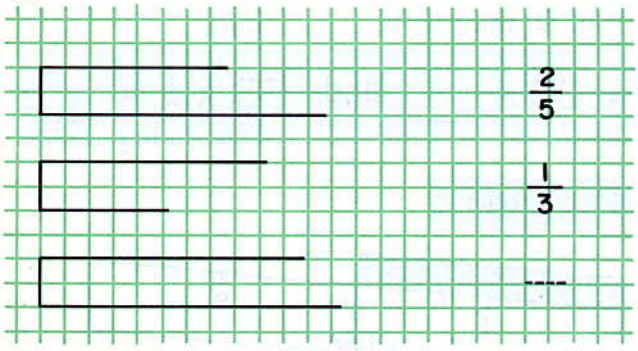
Jean pointed out right away, "Well, I would tell more of your story by writing:

$$\frac{1}{3} + \frac{4}{7} = \frac{7}{21} + \frac{12}{21} = \frac{19}{21}$$

"I like the sketch idea and I also like Jean's way of reporting the results. Let's try some more," Al suggested. Everyone agreed.

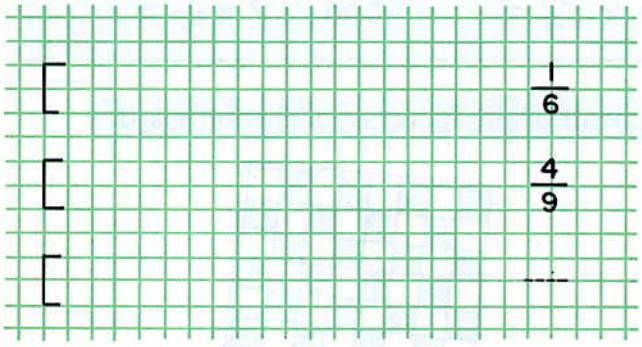
Mrs. Brown provided everyone with graph paper and suggested these problems. Please finish the sketches and the arithmetic they suggest:

A.



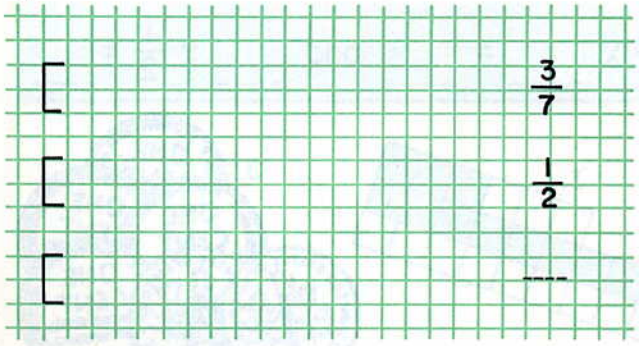
$\frac{2}{5} + \frac{1}{3} = \text{—} + \text{—} = \text{—}$

B.



$\frac{1}{6} + \frac{4}{9} = \text{—} + \quad =$

C.



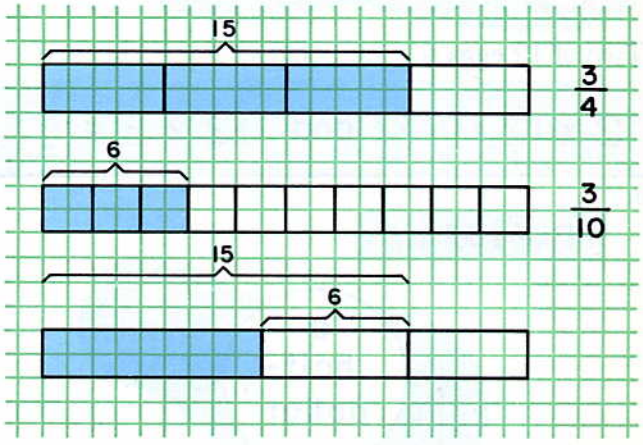
$\frac{3}{7} + \frac{1}{2} = \quad + \quad =$

"I like this method even better than before," Al exclaimed. "I can do any addition and subtraction problems without thinking of a special set of Ben's cards."

"How do you do subtraction?" Alice wanted to know . . . "like $\frac{3}{4} - \frac{3}{10} = \text{\_\_\_\_}$."

"Just draw it out," Al said — and proceeded to show a sketch.

D.



$\frac{3}{4} - \frac{3}{10} = \frac{15}{20} - \text{—} = \text{—}$

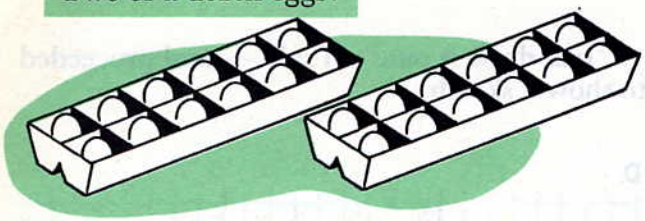
Mrs. Brown urged her class to use whatever method they felt surest with. She gave them some examples to try:

1. $\frac{1}{6} + \frac{3}{4} = \text{—} + \text{—} = \text{—}$
2. $\frac{1}{7} + \frac{2}{3} =$
3. $\frac{5}{9} + \frac{1}{6} =$
4. $\frac{2}{3} - \frac{1}{4} =$
5. $\frac{7}{8} - \frac{1}{2} =$
6. $\frac{1}{4} + \frac{2}{7} =$

Familiar situations — and the arithmetic they suggest.

A.

Two of a dozen eggs:

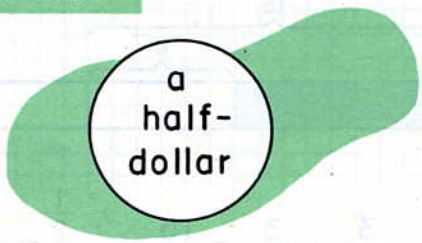


Two of 12 eggs is _____ eggs.

$2 \times 12 = \_\_\_\_$ and $12 \times 2 = \_\_\_\_$

B.

Half a dollar:

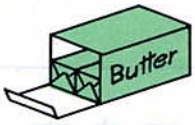


Half of 100 cents is _____ cents.

$\frac{1}{2} \times 100 = \_\_\_\_$ and $100 \times \frac{1}{2} = \_\_\_\_$

C.

Half a pound:

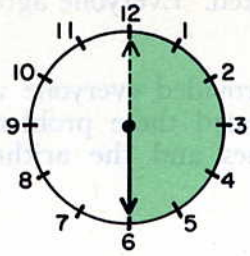


Half of 16 ounces is _____ ounces.

$\frac{1}{2} \times 16 = \_\_\_\_$ and $16 \times \frac{1}{2} = \_\_\_\_$

D.

Half an hour:

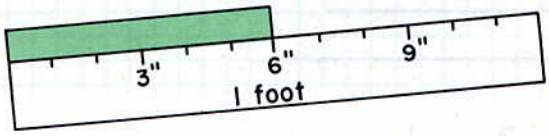


Half of 60 minutes is _____ min.

$\frac{1}{2} \times 60 = \_\_\_\_$ and $60 \times \frac{1}{2} = \_\_\_\_$

E.

Half a foot:

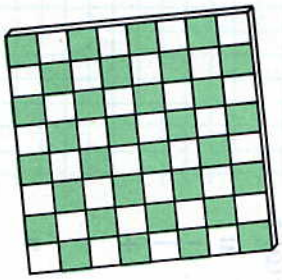


Half of 12 inches is _____ inches.

$\frac{1}{2} \times 12 = \_\_\_\_$ and $12 \times \frac{1}{2} = \_\_\_\_$

F.

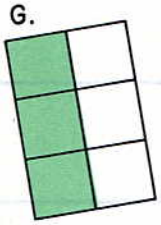
Half of the spaces on a checkerboard:



Half of _____ spaces is _____ spaces.

$\frac{1}{2} \times \_\_\_\_ = \_\_\_\_$ and $\_\_\_\_ \times \frac{1}{2} = \_\_\_\_$

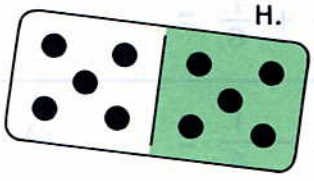
G.



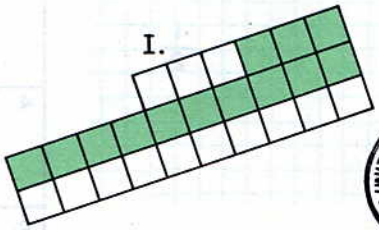
$\frac{1}{2} \times 6 = \_\_\_\_$ and $\_\_\_\_ \times \_\_\_\_ = \_\_\_\_$

$\frac{1}{2} \times 10 = \_\_\_\_$ and $\_\_\_\_ \times \_\_\_\_ = \_\_\_\_$

H.



I.



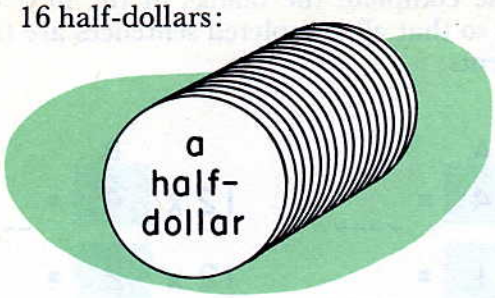
$\frac{1}{2} \times \_\_\_\_ = 13$ and $\_\_\_\_ \times \_\_\_\_ = \_\_\_\_$

$\frac{1}{2} \times 24 = \_\_\_\_$ and $\_\_\_\_ \times \_\_\_\_ = \_\_\_\_$

J.

A.

16 half-dollars:

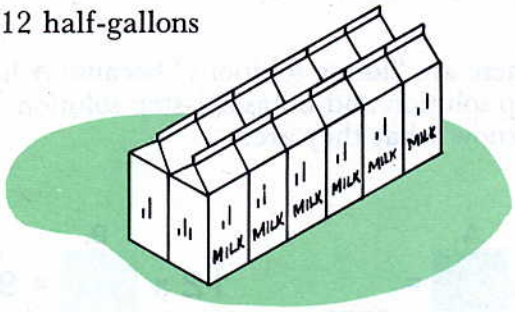


16 half-dollars is _____ dollars

$16 \times \frac{1}{2} =$ ____ and $\frac{1}{2} \times 16 =$ ____

B.

12 half-gallons

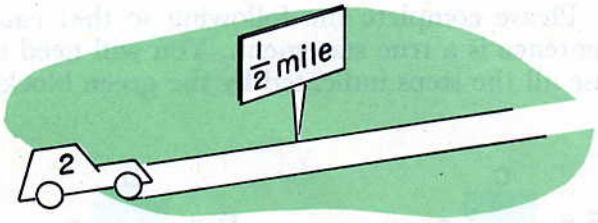


12 half-gallons is _____ gallons

$12 \times \frac{1}{2} =$ ____ and _____ x _____ = _____

C.

100 half-miles:

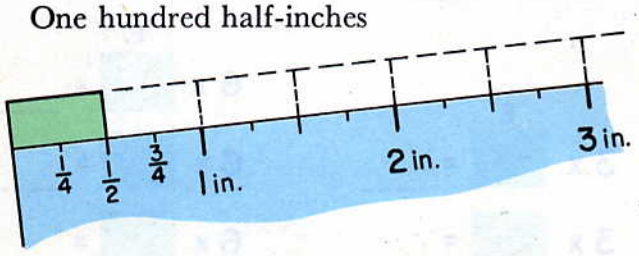


100 half-miles is _____ miles

$100 \times \frac{1}{2} =$ ____ and _____ x _____ = _____

D.

One hundred half-inches

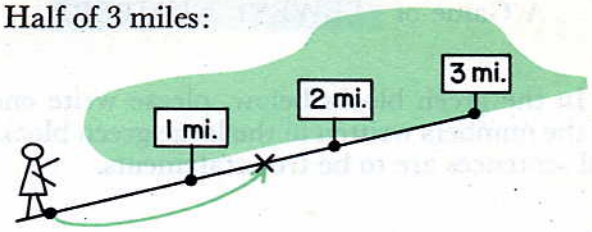


100 half-inches is _____ inches

100 x _____ = ____ and _____ x _____ = _____

E.

Half of 3 miles:

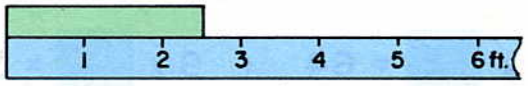


Half of 3 miles is a mile and a half.

$\frac{1}{2} \times 3 = 1\frac{1}{2}$ and $3 \times \frac{1}{2} =$ ____

F.

Half of 5 feet:

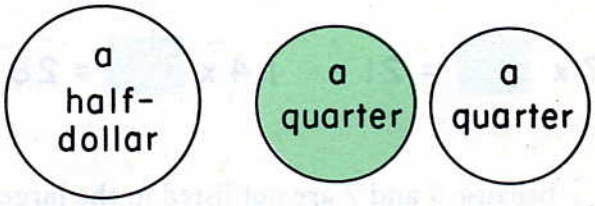


Half of 5 feet is two and a half feet.

$\frac{1}{2} \times 5 =$ ____ and _____ x $\frac{1}{2} =$ _____

G.

Half of a half-dollar:

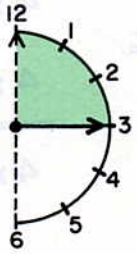


Half of a half-dollar is a quarter.

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

H.

Half of a lunch hour:



Half of a half-hour is a quarter of an hour.

$\frac{1}{2} \times \frac{1}{2} =$ ____

A Game of FEWEST NUMBERS

In the green blocks below, please write one of the numbers written in the large green block. All sentences are to be true statements.

0 , 1 , 2 , 4 , 8 , 16

a. 5 x 4 = 20 b. 10 x ☐ = 80

c. 3 x ☐ = 48 d. 7 x ☐ = 28

e. 8 x ☐ = 64 f. 9 x ☐ = 36

g. 14 x ☐ = 14 h. 17 x ☐ = 0

You cannot complete the following if you follow the rules

i. 7 x ☐ = 21 j. 4 x ☐ = 28

. . . because 3 and 7 are not listed in the large green block above. In such cases, we use a 2- or 3-step solution:

k.
7 x 2 = 14
7 x 1 = ____
7 x 3 = 21

l.
4 x 4 = 16
4 x 2 = ____
4 x 1 = ____
4 x ____ = 28

In such cases, the purpose of the game is to use the fewest numbers from the large green block — to find the solution with the fewest steps.

Please complete the blanks in the next examples so that all completed sentences are true statements:

A.
8 x 4 = ____
8 x 1 = ____
8 x ☐ = ____
8 x 9 = 72

B.
12 x 4 = ____
12 x 2 = ____
12 x ☐ = ____
12 x 8 = 96

These are "losing solutions" because A has a 2-step solution and B has a 1-step solution. Do you know what they are?

A.
8 x ☐ = ____
8 x ☐ = ____
8 x ____ = 72

B.
12 x ☐ = 96

Please complete the following so that each sentence is a true statement. You will need to use all the steps indicated by the green blocks.

C.
5 x ☐ = ____
5 x ☐ = ____
5 x 10 = ____

D.
9 x ☐ = ____
9 x ☐ = ____
9 x 6 = ____

E.
3 x ☐ = ____
3 x ☐ = ____
3 x ☐ = ____
3 x 21 = ____

F.
6 x ☐ = ____
6 x ☐ = ____
6 x ☐ = ____
6 x ☐ = ____
6 x ____ = 90

Another Version of **FEWEST NUMBERS**

The changes in the rules are:

(1) Numbers for the blue blocks must be selected from the large blue block.

(2) No number can be selected more than once in any example.

0 , 1 , 2 , 3 , 6 , 9 , 18

When you do not need one of the steps use "0" in the blue block, and that step will not count as a step.

1

8 x 9 = _____

8 x ☐ = _____

8 x ☐ = _____

8 x 13 = _____

2

9 x 6 = _____

9 x 1 = _____

9 x 0 = _____

x ☐ = 63

A 2-step solution

3

7 x ☐ = _____

7 x ☐ = _____

7 x ☐ = _____

7 x ☐ = 77

This is a 2-step solution

4

8 x ☐ = _____

8 x ☐ = _____

8 x ☐ = 96

5

5 x ☐ = _____

5 x ☐ = _____

5 x ☐ = 75

6

6 x ☐ = _____

6 x ☐ = _____

6 x ☐ = _____

6 x ☐ = 84

7

12 x ☐ = _____

12 x ☐ = _____

12 x ☐ = 72

This is a one-step solution

8

2 x ☐ = _____

2 x ☐ = _____

2 x ☐ = _____

2 x ☐ = _____

2 x ☐ 62

9

13 x ☐ = _____

13 x ☐ = _____

13 x ☐ = _____

13 x ☐ = _____

13 x ☐ 221

10

15 x ☐ = _____

15 x ☐ = _____

15 x ☐ = _____

15 x ☐ = 240

11

3 x ☐ = _____

3 x ☐ = _____

3 x ☐ 72

12

4 x ☐ = _____

4 x ☐ = _____

4 x ☐ = _____

4 x ☐ = 132

13

21 x ☐ = _____

21 x ☐ = _____

21 x ☐ = 252

14

5 x ☐ = _____

5 x ☐ = _____

5 x ☐ = _____

5 x ☐ = 95

15

15 x ☐ = _____

15 x ☐ = _____

15 x ☐ = _____

15 x ☐ = 195

16

10 x ☐ = _____

10 x ☐ = _____

10 x ☐ = _____

10 x ☐ = _____

10 x ☐ = 320

17

9 x ☐ = _____

9 x ☐ = _____

9 x ☐ = _____

9 x ☐ = _____

9 x ☐ = 225

A Third Version of FEWEST NUMBERS

(1) Numbers for the green blocks must be selected from the large green block.

(2) No number (except 0) in this list can be used more than once in any example.

0 , 1 , 2 , 3 , 4 , 5 , 10 , 15
20 , 25 , 50 , 75 , 100 , 125

1 | 2

14 x ☐ = _____
14 x ☐ = _____
14 x 13 = 182

8 x ☐ = _____
8 x ☐ = _____
8 x ☐ = 136

3 | 4

5 x ☐ = _____
5 x ☐ = _____
5 x ☐ = _____
5 x 118 = 590

18 x ☐ = _____
18 x ☐ = _____
18 x ☐ = _____
18 x 23 = 414

5 | 6

10 x ☐ = _____
10 x ☐ = _____
10 x ☐ = _____
10 x ☐ = 470

23 x ☐ = _____
23 x ☐ = _____
23 x ☐ = _____
23 x 14 = _____

7 | 8

8 x ☐ = _____
8 x ☐ = _____
8 x ☐ = _____
8 x 89 = 712

31 x ☐ = _____
31 x ☐ = _____
31 x ☐ = _____
31 x ☐ = 217

9 | 10

24 x ☐ = _____
24 x ☐ = _____
24 x ☐ = 312

15 x ☐ = _____
15 x ☐ = _____
15 x ☐ = 135

11 | 12

19 x ☐ = _____
19 x ☐ = _____
19 x 21 = 399

8 x ☐ = _____
8 x ☐ = _____
8 x 65 = _____

13 | 14

15 x ☐ = _____
15 x ☐ = _____
15 x ☐ = 195

13 x ☐ = _____
13 x ☐ = _____
13 x 19 = 247

15 | 16

25 x ☐ = _____
25 x ☐ = _____
25 x ☐ = 675

27 x ☐ = _____
27 x ☐ = _____
27 x ☐ = 783

17 | 18

30 x ☐ = _____
30 x ☐ = _____
30 x ☐ = 450

x ☐ = 155
x ☐ = 93
31 x ☐ = 248

19 | 20

12 x ☐ = _____
12 x ☐ = _____
12 x ☐ = _____
12 x ☐ = 588

42 x ☐ = _____
42 x ☐ = _____
42 x ☐ = _____
42 x ☐ = 966

A Fourth Version of FEWEST NUMBERS

(1) Numbers for the blue blocks must be selected from the three large blue blocks.

(2) Only one number (except 0) can be used from each list in each example.

(3) Use 0 as often as you can, to show you used less blocks than are provided.

List A. 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

List B. 0, 10, 20, 30, 40, 50, 60, 70, 80, or 90.

List C. 0, 100, 200, 300, 400, . . . 800 or 900.

Remember: You must not use more than one number from a list in each example.

1

12 x ☐ = _____

12 x 3 = _____

12 x ☐ = 156

2

9 x ☐ = _____

9 x ☐ = _____

9 x ☐ = 162

3

15 x ☐ = _____

15 x ☐ = _____

15 x ☐ = 255

4

13 x ☐ = _____

13 x ☐ = _____

13 x ☐ = 312

5

7 x ☐ = _____

7 x ☐ = _____

7 x ☐ = 756

6

6 x ☐ = _____

6 x ☐ = _____

6 x ☐ = 840

7

26 x ☐ = _____

26 x ☐ = _____

26 x ☐ = 390

8

15 x ☐ = _____

15 x ☐ = _____

15 x 20 = _____

9

18 x ☐ = _____

18 x ☐ = _____

___ x ☐ = 324

10

13 x ☐ = _____

13 x ☐ = _____

x 43 = _____

11

24 x ☐ = _____

24 x ☐ = _____

___ x ☐ = 864

12

27 x ☐ = _____

27 x ☐ = _____

27 x ☐ = 513

13

17 x ☐ = _____

17 x ☐ = _____

___ x ☐ = 1870

14

34 x ☐ = _____

34 x ☐ = _____

34 x 71 = _____

15

12 x ☐ = _____

12 x ☐ = _____

12 x ☐ = _____

12 x ☐ = 1524

16

37 x ☐ = _____

37 x ☐ = _____

37 x ☐ = _____

37 x ☐ = 2553

(continued on next page)

Please do the examples below "in your head."

	A.	B.	C.	D.
17	18 x 10 = _____	27 x 4 = _____	27 x 40 = _____	27 x 400 = _____
18	37 x 10 = _____	43 x 8 = _____	43 x 80 = _____	43 x 800 = _____

(continued from previous page)

A. 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

B. 0, 10, 20, 30, 40, 50, 60, 70, 80, or 90.

C. 0, 100, 200, 300, 400, . . . 800, or 900.

Remember: You must not use more than one number from a list in any one example.

1

25 x ☐ = _____

25 x ☐ = _____

25 x ☐ = 425

2

17 x ☐ = _____

17 x ☐ = _____

17 x ☐ = 425

3

9 x ☐ = _____

9 x ☐ = _____

9 x ☐ = _____

9 x ☐ = 1332

4

13 x ☐ = _____

13 x ☐ = _____

13 x ☐ = _____

13 x 267 = _____

5

4 x ☐ = _____

4 x ☐ = _____

4 x ☐ = _____

4 x ☐ = 3516

6

16 x ☐ = _____

16 x ☐ = _____

16 x ☐ = _____

16 x ☐ = 9744

7

36 x ☐ = _____

36 x ☐ = _____

36 x 540 = _____

8

18 x ☐ = _____

18 x ☐ = _____

18 x ___ = 954

Bill's Plan

Bill told Mrs. Brown he used a plan in tackling these problems. She asked him to explain it to the class.

"Well, my first step is to look at the bottom line. For example in Exercise 8 it is:

18 x ? = 954

"Then I think about it this way:

18 x ? = 954

18 x ☐ = _____

"and I look at the largest number in the lists for the blue block so the product won't be more than 954. In this case, it's 50. So I multiply 18 by 50 and subtract to find out how much of the 954 I still need to work on.

18 x ? = 954

− 18 x 50 = 900

54

"Now the problem is cut down in size. The rest of the work is much easier.

"Here are some examples for you to try — using the FIRST STEP of my plan."

A.

24 x ___ = 528

− 24 x 20 = _____

48

_____

B.

15 x ___ = 255

− 15 x ☐ = _____

_____

(There are more examples of Bill's first step on the next page.)

Please do the examples below "in your head." Did you find it convenient to do them mentally?

	C.	D.	E.	F.
9	49 x 20 = _____	35 x 200 = _____	17 x 80 = _____	38 x 400 = _____
10	83 x 30 = _____	72 x 400 = _____	24 x 90 = _____	79 x 800 = _____

1 | 2

23 x ? = 345

23 x ☐ = _____

_____

72 x ? = 2,664

72 x ☐ = _____

_____

3 | 4

105 x ? = 2,625

105 x ☐ = _____

_____

215 x ? = 3,225

215 x ☐ = _____

_____

5 | 6

87 x ? = 8,874

87 x ☐ = _____

_____

39 x ? = 8,190

39 x ☐ = _____

_____

7 | 8

48 x ? = 432

48 x ☐ = _____

_____

39 x ? = 1,404

39 x ☐ = _____

_____

Second Step in Bill's Plan

"After I've reduced the size of the problem I'm usually not done.

"So, I look at what's left in this way, using the last problem as an example:

39 x ? = 1,404

39 x 30 = 1,170

234

39 x ☐ = _____

_____

"Again, I look for the largest number in the lists for the second blue block so the product won't be more than 234. In this case it's 6. So I multiply 39 × 6 and subtract

39 x ? = 1,404

39 x 30 = 1,170

234

39 x ☐ = 234

0

"This problem is finished and I know that

39 x 36 = 1,404

Here are exercises for you to solve using Step One and Step Two of Bill's plan:

9 | 10

15 x ? = 345

15 x ☐ = _____

_____

15 x ☐ = _____

_____

43 x ? = 4,558

43 x ☐ = _____

_____

43 x ☐ = _____

_____

11 | 12

51 x ? = 19,380

51 x ☐ = _____

_____

51 x ☐ = _____

_____

125 x ? = 3,125

125 x ☐ = _____

_____

125 x ☐ = _____

_____

13 | 14

23 x ? = 3,965

23 x ☐ = _____

_____

23 x ☐ = _____

_____

17 x ? = 3,995

17 x ☐ = _____

_____

17 x ☐ = _____

_____

These last two exercises aren't completely solved yet . . . see the next page.

(continued from previous page using same 3 lists)

A. 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

B. 0, 10, 20, 30, 40, 50, 60, 70, 80, or 90.

C. 0, 100, 200, 300, 400, . . . 800, or 900.

"The last two problems weren't completed after the first and second steps," Bill explained. "So, I simply add another step . . . and that is the end!" (Why?)

Here are some examples using Bill's 3-step plan:

1	2
9 x ? = 1,287	25 x ? = 8,175
9 x ▢ = ____	25 x ▢ = ____
9 x ▢ = ____	25 x ▢ = ____
9 x ▢ = ____	25 x ▢ = ____
3	4
38 x ? = 8,322	118 x ? = 18,998
38 x ▢ = ____	118 x ▢ = ____
38 x ▢ = ____	118 x ▢ = ____
38 x ▢ = ____	118 x ▢ = ____

Mrs. Brown helped Bill explain his plan to the class.

Soon, everyone was able to go through the steps until the problems were completed.

Then, Mrs. Brown told the class that this plan was very much like what she called a "kind of long division."

"The only difference," she said, "is that I keep track of my arithmetic in a little different way."

Bill's Plan | Mrs. Brown's Method

A

Bill's Plan	Mrs. Brown's Method
9 x ? = 1,287	9 x ? = 1,287
9 x ▢ = ____	x ▢
9 x ▢ = ____	x 40
9 x ▢ = ____	x 100

```
        1 0 0
9 | 1,2 8 7
    9 0 0
      3 8 7
    3 6 0
        2 7
        2 7
          0
```

B

Bill's Plan	Mrs. Brown's Method
27 x ? = 7,101	27 x ? = 7,101
27 x ▢ = ____	▢
27 x ▢ = ____	▢
27 x ▢ = ____	200

```
       2 0 0
27 | 7,1 0 1
   5 4 0 0
     1
```

Try Bill's plan and Mrs. Brown's method on these examples:

C

32x____ = 768

32x ■ = _____

_____

32x ■ = _____

_____

32x____ = 768

32 | 7 6 8

D

16x____ = 1,968

16x ■ = _____

_____

16x ■ = _____

_____

16x ■ _____

_____

16x____ = 1,968

16 | 1,9 6 8

E

24x____ = 7,416

24x ■ = _____

_____

24x ■ = _____

_____

24x____ = 7,416

24 | 7,4 1 6

F

39x____ = 2,730

39x ■ = _____

39x____ = 2,730

39 | 2,7 3 0

"Let me show you a short cut in the method I've been using," said Mrs. Brown.

"I don't have to use several blue blocks as I have been doing. I notice that all the numbers in each list (except 0) have a first digit from 1 to 9 . . . and the rest of the digits are 0's. So, I usually use the following short cut:

1 2

48x____ = 768

1

48 | 7 6 8

4 8 0

2 8 8

35x____ = 840

2

35 | 8 4 0

"And I can leave out the green band since it simply restates the problem."

3 4

4

63 | 2,5 8 3

15 | 1,3 0 5

5 6

1

35 | 4,9 7 0

3 5 0 0

43 | 8,9 0 1

Following Directions

1. Multiples of 2 are: 0 , 2 , 4 , ___ , ___ , ___ , ___ , ___ , ___ , etc.

2. Multiples of 3 are: ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , etc.

3. Multiples of 5 are: ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , etc.

4. Multiples of 7 are: ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , etc.

In the examples below, list 12 numbers that fit the descriptions given. Write the numbers in order — from the smallest to the largest.

5. Multiples of 2 that are also multiples of 3:

0 , 6 , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___

6. Multiples of 3 that are also multiples of 4:

___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___

7. Multiples of 3 that are also multiples of 5:

___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___

8. Multiples of 2 that are neither multiples of 3 nor multiples of 5:

___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___

9. Multiples of 3 that are neither multiples of 2 nor multiples of 5:

___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___

10. Multiples of 5 that are neither multiples of 2 nor multiples of 3:

___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___

Following Directions

Mark the correct answer. Mark the furthest from correct answer.

How many days are there in a week? (1) 4 (2) 7 (3) 28 (4) 30

(1) (2) (3) (4) 23 ← The difference between the answers you marked.

1

Mary can stay three quarters of an hour. How many minutes are there in three quarters of an hour?

(1) 40 (2) 45 (3) 25 (4) 50

(1) (2) (3) (4) _____

2

Apples are 7 for 25¢. How many could you buy for $3.00?

(1) 21 (2) 12 (3) 73 (4) 84

(1) (2) (3) (4) _____

3

Children's tickets are 45¢ and adults' tickets are 80¢. How much will 3 adult and 4 children's tickets cost?

(1) $4.20 (2) $2.50 (3) $4.55 (4) $1.25

(1) (2) (3) (4) $_____

4

Harry bought a model boat for 67¢ and a model plane for 88¢. He gave the clerk a 5-dollar bill. How much change should he get back?

(1) $2.75 (2) $1.55 (3) $2.45 (4) $3.45

(1) (2) (3) (4) $_____

5

"We've gone one fourth of the way to Aunt Helen's," Mr. Howard said. "We've driven 70 miles. How many miles is it from home to Aunt Helen's house?"

(1) 110 (2) 280 (3) 350 (4) 265

(1) (2) (3) (4) _____

6

The play starts at 7:15 P.M. We need 15 minutes to pick up Jane, 55 minutes to travel, and 10 minutes to park and walk to the theatre. When should we leave?

(1) 6:35 (2) 4:50 (3) 4:30 (4) 5:55

(1) (2) (3) (4) ____hr., ____min.

7

How many quarts will we need for 28 people if we need 2 quarts for each group of 7 people?

(1) 8 (2) 4 (3) 14 (4) 35

(1) (2) (3) (4) _____

8

On a map there is a note:

"Scale: 1 inch = 6 miles"

A length of $2\frac{1}{2}$ inches on the map represents how many miles?

(1) 12 (2) 15 (3) 19 (4) $8\frac{1}{2}$

(1) (2) (3) (4) _____

9

Mrs. Jones bought 6 pounds of fruit. Her correct change from a dollar bill was 46¢. What was the per pound cost of the fruit purchased?

(1) 6¢ (2) 40¢ (3) 9¢ (4) 15¢

(1) (2) (3) (4) _____¢

Letter Patterns

It was a rainy Saturday afternoon.

"Let's hunt for patterns in the letters of words," Numa suggested.

"A pattern in letters of words?"

"Yes, Numo. I pick a Pattern Indicator:

I **a b b c**

That means I want to find 4-letter words with the two middle letters alike," Numa explained.

"Fine," Numo agreed. "You keep a list as we think of them."

meet	room	
boot		

(Please fill in the blanks with your own words that fit the Pattern Indicator.)

"Numa, if I can use names, then 'Anne' would fit the Pattern Indicator."

"Fine, Numo, and you can use plurals, too — 'inn' wouldn't fit, but 'inns' fits the Pattern Indicator."

"How about 'deed?' The middle letters are alike, but the first is the same as the last."

"That's all right, too! Pattern Indicators only tell you the number of letters and which ones must be alike.

"Let's change the Pattern Indicator to:

II **r s t r**

Four-letter words with the same first and last letters."

that	lull	sits
else		

"I can make the hunt more difficult," Numa told her brother, "if I combine the Pattern Indicators. We already have one. Can we find two more?"

III **z m m z**

deed		

They succeeded, but it was a long search.

"Let's try an easier one. My Pattern Indicator is:

IV **a b c c**

"Let's see if we can find 15 examples."

tree	miss	lull
muff		
buzz		

"That was easy. Next, let's find 5-letter words in which the last two letters are the same.

V **a b c d d**

three	bless	
class		
stiff		

"Next, try 5-letter words with the third and fourth letters alike."

VI **g r s s t**

sleep	broom	merry
pills		

Numo and Numa made up a game for their friends. They wrote out a list of Pattern Indicators and a list of words.

Pattern Indicators

(1) abac	(2) fqqfh
(3) ghh	(4) abcab
(5) qrsrt	(6) wxyzxyz
(7) abbea	(8) abccda
(9) fgmqqmf	(10) abacad
(11) cdcecfc	(12) abcbda
(13) wxyzxw	(14) abbcb
(15) qrstqr	(16) xxyzy
(17) abcdceab	(18) abcaad
(19) pqqprspt	(20) xyxx

Players take turns. From the list below, each person tries to find a word that fits a Pattern Indicator in the list above. He writes the number of the Pattern Indicator next to the word and scores a point. If he can't find one, he loses his turn.

( ) seeds	( ) seven
( ) none	( ) fluffy
( ) lull	( ) spools
( ) serves	( ) tooth
( ) sense	( ) headache
( ) eleven	( ) too
( ) llama	( ) geese
( ) states	( ) Alabama
( ) Illinois	( ) dressed
( ) church	( ) singing

Did you match the Pattern Indicators with the words?

Maybe you can do as Numa and Numo did. Make up your own Pattern Indicators and words to fit them.

Here's another way to play a game of Pattern Indicators and words. A player makes up a Pattern Indicator based on a word he knows. The other player tries to find a word to fit the Pattern Indicator. If he does, he gets a point. If he can't, he says, "challenge." Then the one who wrote the Pattern Indicator must give a word that fits. If he can't, he loses a point.

Practice in the examples below. Find words to fit the Pattern Indicators.

1. A abca B tent

2. B pqprst A sister

3. A wxyxz B

4. B amar A

5. prqqp

6. abcdded

7. abcddef

8. axdrengq

9. abccae

10. pdpqbbr

11. abcdac

12. abacdcca

surprise!

I. THINK-OF-A-NUMBER!

Add 3 to it.

Double your result.

Subtract 4.

Divide your result by 2.

Subtract the number you thought of first.

Ahah! Your result is 1.

Try this with your friends.

If they get a final result other than 1, tell them that they must have made a mistake.

Will that surprise them!

You will learn to make up your own "think of a number" games.

II. Here is another "think of a number" game. Try four different examples.

	Examples: A	B	C	D
Think of a number.	5	8	100	
Add 4.	9	12		
Multiply by 3.	27			
Subtract the number you thought of first.				
Divide by 2.				
Subtract 2.				
Subtract the number you thought of first.				

III. Complete the columns below:

a.	1 x 9 =	0	9	0 + 9 =	____
b.	2 x 9 =	1	8	1 + 8 =	____
c.	3 x 9 =	2	7	2 + 7 =	____
d.	4 x 9 =		6	+ 6 =	____
e.	5 x 9 =			+ =	____
f.	6 x 9 =			+ =	____
g.	7 x 9 =			+ =	____
h.	8 x 9 =			+ =	____
i.	9 x 9 =			+ =	____
j.	10 x 9 =			+ =	____

IV.

Do the multiplications before doing the subtractions (as in the first example below).

a.	1 x 2	− 0 x 3	=	2	− 0	=	2
b.	2 x 3	− 1 x 4	=	6	−	=	____
c.	3 x 4	− 2 x 5	=		−	=	____
d.	4 x 5	− 3 x	=		−	=	____
e.	5 x 6	− 4	=		−	=	____
f.	6 x 7	−	=		−	=	____
g.	7 x	−	=		−	=	____
h.	8	−	=		−	=	____
i.	9	−	=		−	=	____

V.

1	2	3	4	5	6	7
x 1	x 2	x 3	x 4	x 5	x 6	x 7
1	4					
	3					
		2				

Carry out the multiplications. Record results in the grey spaces.

In each blue space, record the difference between the two numbers written above the blue space.

VI.

2	3	4	5	6	7	8
x 0	x 1	x 2	x 3	x 4	x 5	x 6
0	3					
	3					
		2				

Use the green spaces similarly. In each, record the difference of the numbers written in the blue spaces above the green space.

Compare the results.

Did you expect what you see?

VII.

a. (0 x 3) ÷ 2 = ____

b. (1 x 4) ÷ 2 = ____

c. (2 x 5) ÷ 2 = ____

d. (3 x 6) ÷ 2 = ____

e. (4 x) ÷ 2 = ____

f. (5 x) ÷ 2 = ____

g. (6 x) ÷ 2 = ____

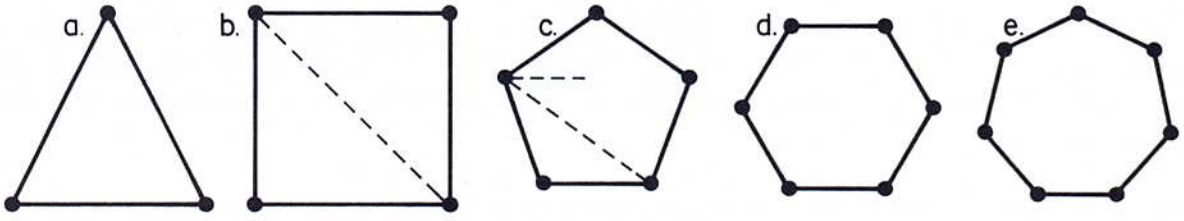
h. (7 x) ÷ 2 = ____

Find the results in the group of examples on the left. Record these results in the grey spaces.

a.	b.	c.	d.	e.	f.	g.	h.
0	2						
	2						

In the blue spaces, record the differences of the numbers in the grey spaces. In the green spaces, record the differences of the numbers in the blue spaces.

VIII. Draw all the diagonals you can connecting the dots in each of the figures below. Record the number of diagonals in the grey spaces.



	a.	b.	c.	d.	e.
Number of diagonals →					
Difference of numbers above →					
Difference of numbers above →					

Have you seen a similar result before?

Anne gave each member of the class a bag and some counters. She said, "Think of a number. Put that number of counters in the bag, and tie it up.

Put 3 counters next to the bag. Now, if you tell me how many counters you have altogether, I will tell you what's in the bag."

Game I

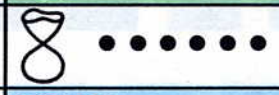
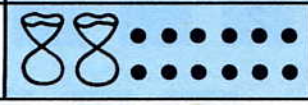
		Betty	Jim	Ruth	Ted	Ben
Think of a number. Put that many counters in the bag.		5				
Put 3 counters next to the bag. How many counters altogether?		8	6	12	17	3

What numbers did Betty, Jim, Ruth, Ted, and Ben think of ?

"That's too easy," Ben complained.

"All right," Anne continued, "you will each need extra bags and plenty of counters for the next games."

Game II

		Betty	Jim	Ruth	Ted	Ben
Think of a number. Put that many counters in the bag.		4				
Add 6. You now have the bag and 6 extra counters.		10				
Double it. You now have two bags and 12 extra counters. How many counters altogether?		20	34	42	46	12

Game III

Think of a number.						
Double it.						
Add 6.						
Divide by 2. How many counters have you altogether?		15	7	3	20	4

Game IV

Think of a number.						
Add 7.						
Add the number you thought of first.						
Subtract 5. How many have you altogether?		16	22	52	2	4

"That game," Ben said, "reminds me of the trick I saw a little while ago. I can make these tricks up myself now. Here is an example:

Think of a number.
Double it. Then add 6.
Divide by 2.
Take away the number you first thought of."

YOUR RESULT IS 3!

Ben explained it this way:

Trick I

Think of a number.		Think of a number. Put that many counters in a bag.
Double it.		Double it. Now you have 2 bags with as many counters in each as the number you thought of first.
Add 6.		Add 6. Now you have 2 bags plus 6 extra counters.
Divide by 2.		Divide by 2. You remove 1 of the bags and 3 of the counters, leaving a bag and 3 counters.
Subtract the number you thought of first.		Now, subtract the number you thought of first.
Your result is 3!		Your secret number disappears from the scene, no matter how many counters were in the bag.

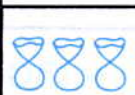
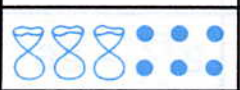
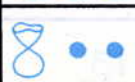
"Here are two more tricks," Ben volunteered. "They are longer, but they are based on the same idea."

The numbers Anne and Bill thought of are shown.
Also try a number you think of yourself.

Trick II

		Anne	Bill	You
Think of a number.		2	8	
Add 3.		5		
Triple the result.				
Subtract 1.				
Subtract the number you thought of first.				
Divide by 2.				
Subtract the number you thought of first.				

Trick III

		Anne	Bill	You
Think of a number.		10	5	
Double it.		20		
Double it again.				
Subtract the number you thought of first.				
Add 6.				
Divide by 3.				
Subtract the number you thought of first.				

Next day, Jerry said that he had made up tricks like Ben's but with larger numbers. Instead of drawing counters, Jerry writes numbers to keep track of the extra counters.

(Below is the number Ben thought of. Try a number you think of.)

Trick IV

		Ben	You
Think of a number.		19	
Add 50.	+ 50		
Double it.	+ 100		
Add 48.	+ 148		
Divide by 2.	+ 74		
Subtract the number you thought of first.			

"Now," Jerry went on, "For the next trick I'm going to have to draw large numbers of bags unless we can agree that all I need to do is draw one bag and tell you how many I mean to draw."

Trick V

		Ben	You
Think of a number.		7	
Multiply by 10.	10		
Add 130.	10 + 130		
Divide by 5.	2 + 26		
Divide by 2.	+ 13		
Subtract the number you thought of first.			

"Those bags are hard to draw," Jerry complained. "So I'm not going to draw the top and bottom of the bag. All I shall draw is an X which looks very much like the letter *x*. So, for example, I write 3X + 6 when I mean 3 bags and 6 extra counters."

Trick VI

	Shorthand	Ben	You
Think of a number.	X	3	
Add 47.	X + 47	50	
Double it.	2X + 94		
Subtract 75.	2X + 19		
Triple it.	6X + 57		
Subtract the number you thought of first.	5X + 57		
Add 18.	5X + 75		
Divide by 5.	X + 15		
Subtract the number you thought of first.			

Finish the picture and use Jerry's shorthand to show the following trick:

Trick VII

Words	Pictures	Shorthand
Think of a number.		X
Add 4.		X + 4
Double it.		
Subtract 2.		
Add the number you thought of first.		
Divide by 3.		
Subtract the number you thought of first.		

"If you show me the result at any stage of the games or tricks, I'll tell you the number the player was thinking of to start with. The only exceptions would be the last lines of the tricks, because there has to be at least one bag left in the picture."

Fred did it. Can you discover his method?

Game II — page 102

		Betty	Jim	Ruth	Ted	Ben
Think of a number.						
2nd line →		10	17	21	23	6

Game III — page 102

Think of a number.						
3rd line →		30	14	6	40	8

Game IV — page 102

Think of a number.						
4th line →		16	22	52	2	4

Trick II — page 103

		Anne	Bill
Think of a number.			
5th line		12	24

Trick III — page 103

		Anne	Bill
Think of a number.			
5th line		36	21

Trick IV — page 104

		Ben	You
Think of a number.			
3rd line	+ 100	138	

Trick V — page 104

		Ben	You
Think of a number.			
4th line	+ 26	40	

Trick VI — page 104

		Ben	You
Think of a number.	X		
7th line	5X + 75	90	

Trick VII — page 104

		Al	Ed
Think of a number.	X		
5th line	3X + 6	18	33

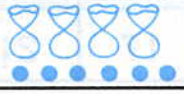
Two Extras

		P	Q
Think of a number.	X		
	4X + 7	11	35

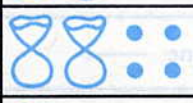
		R	S
Think of a number.	X		
	3X + 150	225	150

Here are incomplete records of some tricks Fred gave to the class. Please complete each record.

I.

Words	Pictures	Examples a	Examples b
Think of a number.			
Multiply by 2.			
Multiply by 2 again.		20	
Add 6.			
Subtract the number you thought of first.			33
Divide by 3.			
Add 8.			
Subtract the number you thought of first.			

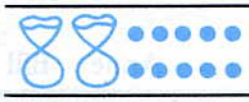
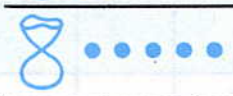
II.

Words	Pictures	Examples a	Examples b
			15
			
		13	

III.

Words	Shorthand	Examples a	Examples b
Think of a number.	x		
Add 9.			
Double.	$2x + 18$		26
Add the number you thought of first.			
Divide by 3.			
Add 10.		26	
Double.			
Subtract the number you thought of first.			
Subtract 25.			
Subtract the number you thought of first.			

IV.

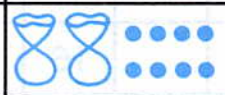
Pictures	Shorthand	Examples a	Examples b
			40
		14	

V.

Pictures	Shorthand	Examples a	Examples b
		4	7
		12	21
		24	33
		20	26
		10	13
		6	6

WANTED: Endings for these tricks.

Trick A

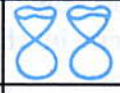
Words	Pictures	Shorthand	Examples a.	Examples b.
Think of a number.		X	9	
Add 4 to it.		X+4		
Double it.		2X+8		
Subtract 2.		2X+6		

Trick C

Words	Shorthand	Example
Think of a number.	X	
Add 6.	X+6	
Double it.		

Can you finish in two or three steps? In the step before the end, there should not be more than a single bag. When you say, "Subtract the number you thought of first," that gets rid of the bag.

Trick B

Words	Pictures	Shorthand	Examples a.	Examples b.
Think of a number.		X		
Double it.		2X		
Add 9.				
Add the number you thought of first.				

Trick D

Words	Shorthand	Example
Think of a number.	X	
Triple it.	3X	
Add 8.		
Subtract the number you thought of first.		

Fred had another idea!

"If you show me part of a record of the kind of tricks and games we've been playing, I can often figure out the rest. You don't have to show me the words or pictures or shorthand.

"Here is a sample of the ways two tricks might get started. I'll explain my idea as we go along."

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
Think of a number.		x	5	3	1	11		0	25
			10	8		16	12	5	

"Consider Anne's results. She thought of 5. By applying some rule not shown, she was led to 10. The rule might be to double the number or, perhaps, to add 5. Ben thought of 3. Doubling would lead to 6. But his result is 8. However, adding 5 would explain the results of both Anne and Ben.

"The rule of adding 5 would also explain the results recorded for Jane and Joe. Applying this rule to the numbers Matt and Betty thought of would lead to 6 and 30. And we can see that the number Ruth thought of would have been 7 because $7 + 5 = 12$.

"Now we can draw the pictures and show the idea in shorthand."

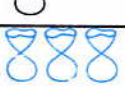
Here are the first steps of another trick:

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
Think of a number.		x	0	100			10	12	25
			0		21	33	30		75
				315				51	90

"From Anne's results in the first 2 rows, I know the rule could be add 0 or it could involve multiplication. Ruth thought of 10 and the rule led her to 30. It must be to multiply by 3 or, in other words, triple the number you thought of.

"Tripling would also explain Betty's results. Now I can draw pictures and write the shorthand for the second row. I can also find out the missing numbers in the first and second lines."

Next, consider the second and third rows.

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
Think of a number.		x	0	100	7	11	10	12	25
Triple it.		3x	0	300	21	33	30	36	75
				315				51	90

"Ben's results suggest right away that the rule might be to add 15. That rule explains the results given for Joe and Betty. Now I can complete the third line."

Here is the record of a trick Fred says that he can reconstruct:

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
a. Think of a number.			11	7	3		1		9
b.			22	14	6			12	
c.							8	18	24
d.			14		6	11	4		
e.				3		3		3	3

"Consider lines a and b," Fred advises.

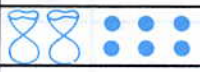
a. Think of a number.			11	7	3		1		9
b.			22	14	6			12	

"In line a, I begin by drawing a bag. When anyone thinks of a number, I suppose that they put that many counters in the bag. Then I use Jerry's shorthand. I erase the top and bottom of the bag. The part that is left looks like the letter *x*.

"Next, I consider the results shown for Anne, Ben, and Matt. The rule is to double the number you thought of. So, I write this idea in words, pictures, and shorthand. Now, I can fill in the missing results for Ruth, Joe, and Betty."

			Anne	Ben	Matt	Jane	Ruth	Joe	Betty
b. Double it.		2X	22	14	6		2	12	18
c.							8	18	24

"By studying the results for Ruth, Joe, and Betty, I know the rule must be to add 6."

c. Add 6.		2X + 6	28	20	12		8	18	24
d.			14		6	11	4		

"By studying the results shown for Anne, Matt, and Ruth, I know the rule is to divide by 2."

a. Think of a number.		X	11	7	3		1	6	9
d. Divide by 2.		X + 3	14	10	6	11	4	9	12
e.				3		3		3	3

"Consider the lines a, d, and e. The results for Ben, Joe, and Betty give the clue. The rule must be to subtract the number you thought of first. Now, you can complete the reconstruction of the entire record."

Ben made up some challenging tricks. Can you reconstruct the records?

I

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
Think of a number.		X	10			5		8	
					12	8		11	
			26	6	24		30		
				10			34		
				5		10			20
				5					5

II

Words	Pictures	Shorthand	Anne	Ben	Matt	Jane	Ruth	Joe	Betty
Think of a number.									
			12	0	30		75		
		3X + 6	18	6	36				
				6	26	16	56		
								103	22
Subtract the number you thought of first.		3							

III

Pictures	A	B	C	D
				8
		17		18
	7	11	15	
	4		4	

V

Pictures	I	J	K	L
	2	4		0
		9	20	5
	5		5	
		1		1

IV

Shorthand	E	F	G	H
X				7
		0		
2X + 20		20	26	34
	26	20		27
			5	
2				

VI

Shorthand	M	N	O	P
X				
	9		13	
3X + 12	27			
		38		34
X + 6		19	15	17
6				

Fred repeats his claim:

"Some of these guessing games are becoming more complicated. But, if you give me the shorthand and the results in any row, I can still tell the number each player thought of first."

	Shorthand	a	b	c	d	e	f	g	h	i
A. One row of results.	X + 8	18	19	20	8	100	16	39	29	9
The number each player thought of first.	X									
B. One row of results.	2X + 5	15	35	55	5	7	9	115	91	67
The number each player thought of first.	X									
C. One row of results.	4X − 3	1	9	17	97	37	397	25	41	57
The number each player thought of first.	X									
D. One row of results.	10 + X	12	20	11	60	10	21	39	35	17
The number each player thought of first.	X									
E. One row of results.	3X − 6	15	0	30	24	9	144	36	69	84
The number each player thought of first.	X									
F. One row of results.	200 − X	195	199	125	200	193	190	50	100	19
The number each player thought of first.	X									
G. One row of results.	25X + 25	225	125	50	275	150	25	400	200	250
The number each player thought of first.	X									
H. One row of results.	9+3X+5X	9	33	17	65	25	49	97	89	169
The number each player thought of first.	X									
I. One row of results.	4X+7−2X	11	7	29	9	21	27	107	57	17
The number each player thought of first.	X									
J. One row of results.	9+2X+5X	65	44	16	58	9	72	86	184	79
The number each player thought of first.	X									

Anne explains an interesting and useful result.

"If I think of any number, I can always get the same results in two ways that seem very different.

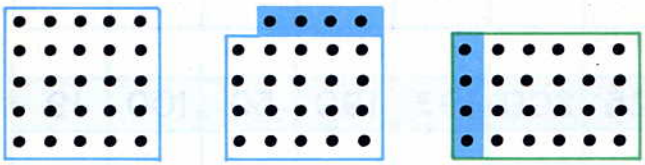
"I'll keep track of these ways in shorthand. I'm not going to use bags or 'χ' because 'n' or any other mark would be just as good, as long as we agree it means the number thought of first."

I. (Try some examples of your own)

Words	Shorthand	Ben	Al	Mary	Jane	A	B	C	D
Think of a number.	n	3	5	11	4				
Multiply your number by itself.	n · n	9							
Subtract 1 from the product.	(n · n) − 1	8							
Add 1 to your number.	n + 1	4							
Subtract 1 from your number.	n − 1	2							
Multiply your results together.	(n + 1) · (n − 1)	8							

"For any number you can think of, is it true that $(n \cdot n) - 1 = (n + 1) \cdot (n - 1)$?"

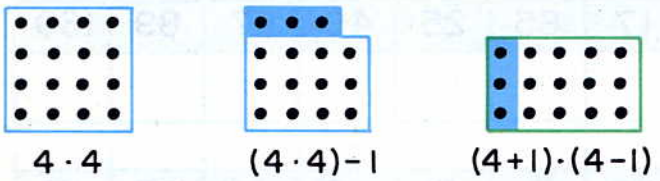
"You can make a picture to show why this happens. For example, the number Al thought of was 5."



5 · 5 (5 · 5) − 1 (5+1) · (5−1)

"After you take one dot away from 5 rows of 5 dots, you have just enough to make 4 rows of 6 dots.

"Jane's example works the same way."



4 · 4 (4 · 4) − 1 (4+1) · (4−1)

(n · n) − 1 = (n+1) · (n−1)

So does Mary's example.

11 · 11 (11 · 11) − 1 = (11+1) · (11−1)

"This is useful as a shortcut. If you want to know the answer to 21×19, you can think of it as $(20 + 1) \cdot (20 - 1)$. You also know that you can find the result by finding $(20 \cdot 20) - 1$. It is 399. Please use this same method in the following:

a. 31 x 29 = _____ d. 51 x 49 = _____

b. 13 x 11 = _____ e. 101 x 99 = _____

c. 41 x 39 = _____ f. 71 x 69 = _____

"It is also a shortcut when you want to find one less than 21×21. You know that $(21 \cdot 21) - 1 = (21 + 1) \times (21 - 1)$ or 22×20. The result is 440."

g. (31 · 31) − 1 = _____ i. (51 · 51) − 1 = _____

h. (41 · 41) − 1 = _____ j. (101 · 101) − 1 = _____

"So, of course, you can use this shortcut in the following:"

k. 31 · 31 = _____ m. 51 · 51 = _____

l. 21 · 21 = _____ n. 41 · 41 = _____

"This result is quite limited as a shortcut," Anne continued. "But it suggests further experiments that lead to similar shortcuts. Let's agree again that 'n' is shorthand for any number you think of. However, any letter you care to use is just as useful."

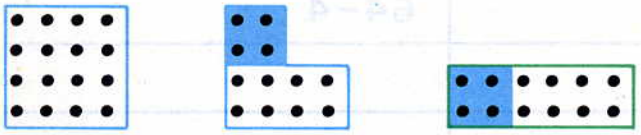
II.

(Examples of your own)

Words	Shorthand	Ben	Al	Mary	Jane	E	F	G	H
Think of a number.	n	4	3	2	5				
Multiply it by itself.	$n \cdot n$	16							
Subtract 4 from the product.	$(n \cdot n) - 4$								
Add 2 to your number.	$n + 2$	6							
Subtract 2 from your number.	$n - 2$	2							
Multiply your results together.	$(n+2) \cdot (n-2)$								

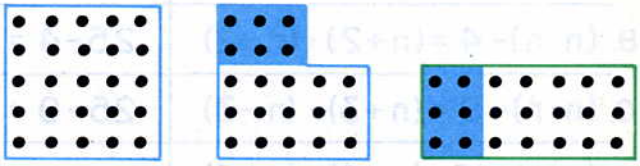
"For any number you can think of, is it true that $(n \cdot n) - 4 = (n + 2) \cdot (n - 2)$?"

"You can draw a picture to show why this happens. For example: Ben thought of 4.



$4 \cdot 4$ $\quad$ $(4 \cdot 4) - 4$ $\quad$ $(4+2) \cdot (4-2)$

"Jane thought of 5. You can draw a picture to show the results."



$5 \cdot 5$ $\quad$ $(5 \cdot 5) - 4$ $\quad$ $(5+2) \cdot (5-2)$

"You can use this result as a shortcut in the examples to the right."

a. $(22 \cdot 22) - 4 =$ _____

b. $(32 \cdot 32) - 4 =$ _____

c. $18 \cdot 22 =$ _____

d. $48 \cdot 52 =$ _____

III.

(Examples of your own)

Words	Shorthand	Ben	Al	Mary	Jane	I	J	K	L
Think of a number.	n	7	4	3	5				
Multilply it by itself.	$n \cdot n$								
Subtract 9 from that product.	$(n \cdot n) - 9$								
Add 3 to your number.	$(n + 3)$								
Subtract 3 from your number.	$(n - 3)$								
Multiply your results together.	$(n+3) \cdot (n-3)$								

"For any number you can think of, is it true that $(n \cdot n) - 9 = (n + 3) \cdot (n - 3)$?"

Anne gave another example:

IV

Words	Shorthand	Ben	Al	Mary	Jane	(Examples of your own) M	N	O	P
Think of any number.	n	5	10	4	8				
Multiply it by itself.	$n \cdot n$								
Subtract 16 from that product.	$(n \cdot n) - 16$								
Add 4 to your number.	$n + 4$								
Subtract 4 from your number.	$n - 4$								
Multiply your results together.	$(n+4) \cdot (n-4)$								

"For any number you can think of, is it true that $(n \cdot n) - 16 = (n + 4) \cdot (n - 4)$?"

"Now, let's summarize our results," Anne suggested. "On this and the last two pages, we have found the following which we can test together."

	The number Ben thought of was 5	The number Mary thought of was 8
A. $(n \cdot n) - 1 = (n+1) \cdot (n-1)$	$25 - 1 = 6 \cdot 4$	$64 - 1 = 9 \cdot 7$
B. $(n \cdot n) - 4 = (n+2) \cdot (n-2)$	$25 - 4 =$	$64 - 4$
C. $(n \cdot n) - 9 = (n+3) \cdot (n-3)$	$25 - 9 =$	
D. $(n \cdot n) - 16 = (n+4) \cdot (n-4)$		

"Then the following group of statements ought to be true. Let's test them."

	If the number thought of is 9	If the number thought of is 4
E. $n \cdot n = (n+1) \cdot (n-1) + 1$	$81 = (10 \cdot 8) + 1$	$16 = (5 \cdot 3) +$
F. $n \cdot n = (n+2) \cdot (n-2) + 4$	$81 = (11 \cdot 7) +$	
G. $n \cdot n = (n+3) \cdot (n-3) + 9$		
H. $n \cdot n = (n+4) \cdot (n-4) + 16$		

"The following statements must also be true," Anne asserted.

	If the number thought of is 7	If the number thought of is 10
I. $(n+1) \cdot (n-1) = (n \cdot n) - 1$	$8 \cdot 6 = 49 -$	
J. $(n+2) \cdot (n-2) = (n \cdot n) - 4$		
K. $(n+3) \cdot (n-3) = (n \cdot n) - 9$		
L. $(n+4) \cdot (n-4) = (n \cdot n) - 16$		

Finally, Anne ties these ideas together into a handy shortcut.

"There is a pattern running through the examples on the last three pages," Anne explained. "It is something like a game that might be called 'Think-of-Any-Two-Numbers.' Here's the way it works."

(Your own examples)

Think-of-Any Two Numbers	Shorthand	Ben	Ruth	Ellen	Roy	A	B	C
The larger number.	b	7	6	10	15			
The smaller number.	s	3	4	7	10			
The sum of your numbers.	b + s	10						
The difference of your numbers.	b − s	4						
The product of the sum and difference of your numbers.	(b+s)·(b−s)	40						
The product of the larger number multiplied by the larger number.	b · b	49						
The product of the smaller number multiplied by the smaller number.	s · s	9						
The difference of the products.	(b · b)−(s · s)	40						

"For any pair of numbers you think of, $(b + s) \cdot (b - s) = (b \cdot b) - (s \cdot s)$."

"Many pairs of factors," Anne explained, "can be rewritten in the form of $(b + s) \cdot (b - s)$ and in the form of $(b \cdot b) - (s \cdot s)$. Often, these forms point to important shortcuts. Sometimes they do not help at all."

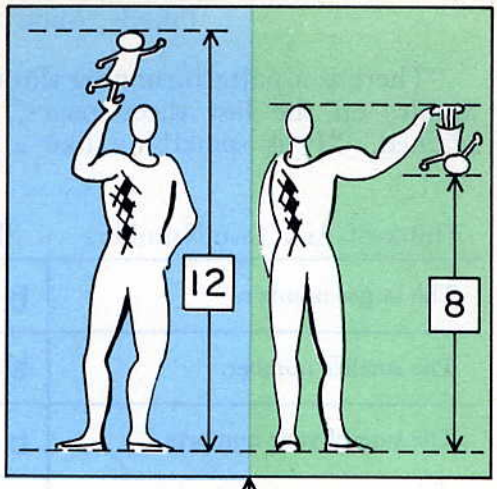
	The product of the pair of factors.	The product rewritten in the form of $(b + s) \cdot (b - s)$ or $(b - s) \cdot (b + s)$	The product rewritten in the form of $(b \cdot b) - (s \cdot s)$		
A.	12 · 8 =	(10+2) · (10−2) =	(10 · 10)−(2 · 2) =	100 − 4 =	_____
B.	10 · 16 =	(13−3) · ( ) =	=	=	
C.	21 · 19 =	(20+1) · ( ) =	=	=	
D.	22 · 18 =	=	=	=	
E.	35 · 25 =	=	=	=	
F.	53 · 47 =	=	=	=	
G.	56 · 44 =	=	=	=	
H.	109 · 91 =	=	=	=	
I.	79 · 81 =	=	=	=	

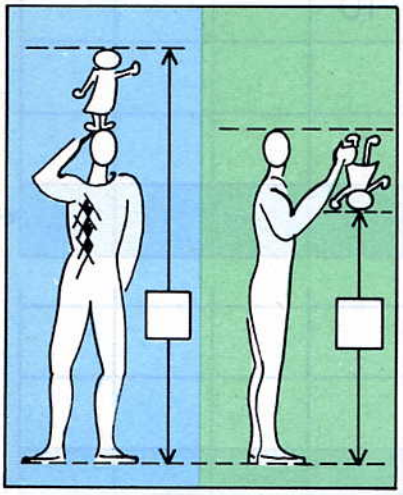
Make up examples of your own.

Guessing Games in the Land of Giant Acrobats and Dwarf Acrobats.

In each team of acrobats, giants are the same height and dwarfs are the same height. (All heights are given in numbers of feet.)

An example: The height of a giant and a dwarf together is 12 feet. A giant's height less that of a dwarf is 8 feet. How tall is each?



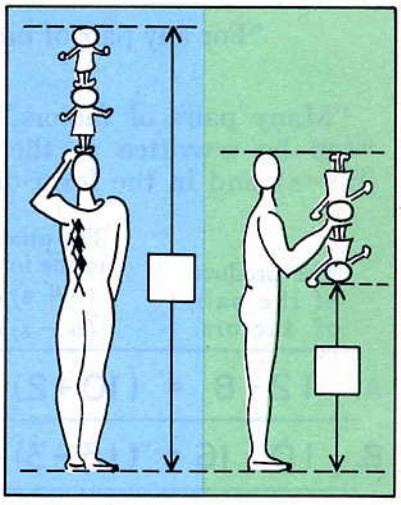


Ben and Anne imagined different results:

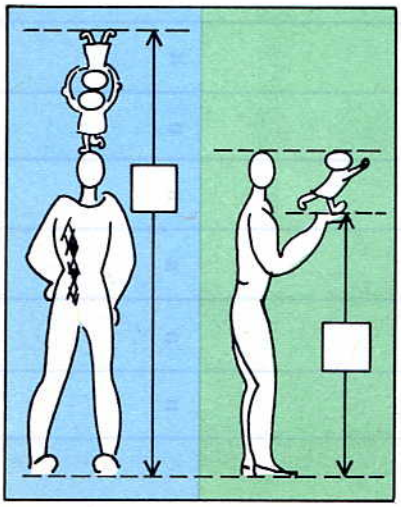
I.

The number of feet in the height of:	Shorthand	Example	Ben	Anne
a giant and a dwarf together	g + d	12		15
a giant less that of a dwarf	g − d	8		13
a giant	g		28	
a dwarf	d		5	

II. Fred, Dot, and Mary each imagined heights for members of another team of acrobats. Please guess the heights each person imagined.



The number of feet in the height of:	Shorthand	Fred	Dot	Mary
a giant and two dwarfs altogether	g + 2d	18		45
a giant less that of two dwarfs	g −		16	25
a giant	g		20	
a dwarf	d	3		



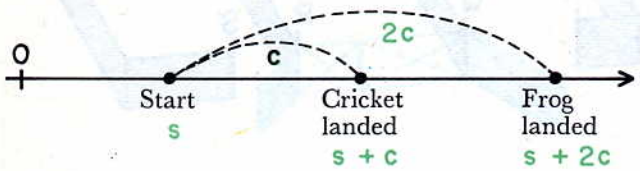
III. Al, Jo, and Nan each imagined other heights for the arrangement of acrobats shown on the left. Please guess the heights each person imagined.

The number of feet in the height of:	Shorthand	Al	Jo	Nan
a giant and two dwarfs altogether	g +	29	50	20
a giant less that of a dwarf	g −	23		14
a giant	g		40	
a dwarf	d			

Ruth's Guessing Game . . . (but she wasn't really guessing)

Ruth drew a number line on the board.

"A cricket and a frog started at the same point. The frog jumped twice as far as the cricket in the direction away from 0.



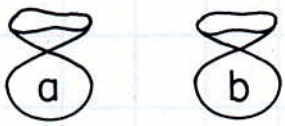
You must decide where each jumper started from and how far the cricket jumped. Tell me where each jumper landed. I'll tell you where you decided the jumpers started from and in what direction each jumped."

I.

	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number that indicates the starting point.	s	5								
Think of another number that indicates the size of the cricket's jump.	c	3								
The frog jumps twice as far as the cricket.	2c	6								
Tell where the cricket landed.	s + c	8	16	14	7	24	12	14	17	71
Tell where the frog landed.	s + 2c	11	22	18	14	34	23	21	26	92

Ruth told each player the numbers the player had thought of. Can you?

Ruth drew a pair of bags and labeled one "a" for Andy, and the other "b" for Bill. Then she explained that Andy had more marbles in his bag than Bill had.



"Suppose you know how many each had. Use any two numbers you think of. Just tell me how many they have altogether and how many more Andy has than Bill."

II.

	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number to indicate how many marbles Andy had.	a	10								
Think of a number to indicate how many marbles Bill had.	b	3								
How many marbles together?	a + b	13	15	1	10	17	28	17	64	132
How many more marbles does Andy have than Bill?	a − b	7	3	1	6	1	12	17	6	34

Again, Ruth told each player the numbers he had thought of. Can you?

(Make up your own story for the next example.)

III.

	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number.	m	3								
Think of a second number.	n	5								
Add the two numbers.	m + n	8	4	6	8	9	2	25	18	30
Add the second to 3 times the first.	3m + n	14	8	16	10	27	4	45	40	60

A Gate Problem

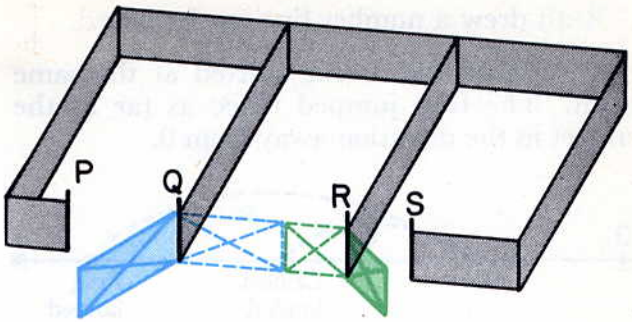
Ruth drew a diagram and explained:

"Each of you pretend you are a farmer with three fields. Each field is fenced in as shown. You have only two gates — one blue and one green. The gates may be of different lengths. You arrange them as shown.

"The blue gate alone can close off the field on the left. The green gate alone can close off the field on the right. The blue gate and green gate are both needed to close off the middle field. In this case, the gates meet but do not overlap.

"In order that we may talk about them, I have labeled four posts that are in a line — P, Q, R, and S. Since you know the length of each gate, you also know how far apart to place the posts.

"In the chart below, fill in *any two* of the questions marked A, B, C, D, E, F.



"Unless you trick me by answering a certain pair of questions, or by not giving me enough information to work on, I can figure out the length of each gate and the distances between the posts.

"Of course, if you answer the first two questions, my job is easy. Can you find a pair of questions that do not give me enough information?"

I.

Some of the players

What is the number of feet in:	Shorthand	Anne	Ben	Carl	Dan	Ellen	Fred	Gil	Harry	Ivan	June
A. the length of the blue gate	b										
B. the length of the green gate	g										
C. the distance from Q to R		11	19		27			19		26	
D. the distance from P to R		18		41			32		28		15
E. the distance from P to S				52	54	52	46	38			
F. the distance from Q to S			28			39			38	41	

Did you discover the two players that stumped Ruth? They were the ones that answered questions C and E.

Compare all the results in row C with those in row E. Do you notice a pattern? Does it help explain why Dan and Gil stumped Ruth?

Did Harry and June give Ruth enough information to work on? _____

If some player answered only question C, would you know how he would have to answer question E? _____

If you were told that the blue gate is twice as long as the green one, can you complete a chart with only 1 question answered?

II.

	Shorthand	K	L	M	N
A	b				
B	g		5		
C	b + g	27			
D	2b + g				
E	2b + 2g				66
F	b + 2g			24	

"Here's another game," Ruth began.

"Each of you think of any two numbers. Then tell me what you get for any two of the six expressions given below.

I will tell you what you would get for the other expressions. I will also tell you the two numbers you thought of."

I.

Player	one number (Shorthand) x	other number (Shorthand) y	x + y	2x + y	x + 2y	3x + y	3x + 2y	3x + 4y
Anne			8	11				
Betty			9			23		
Carl					15		25	
Dan			19		28			
Ellen				31		46		
Fred				19			34	
Gil						10		22

"Think of any two numbers (one larger than the other). The rest of the rules are the same. I will change the expressions. Please tell me what results you get for any two expressions.

II.

Player	bigger number (Shorthand) b	smaller number (Shorthand) s	b + s	b − s	b + 2s	2b − s	2b + s	3b − s
Anne			15	5				
Betty				1		10		
Carl				3	6			
Dan						22	26	
Ellen						25		40
Fred			6					18
Gil			11				18	

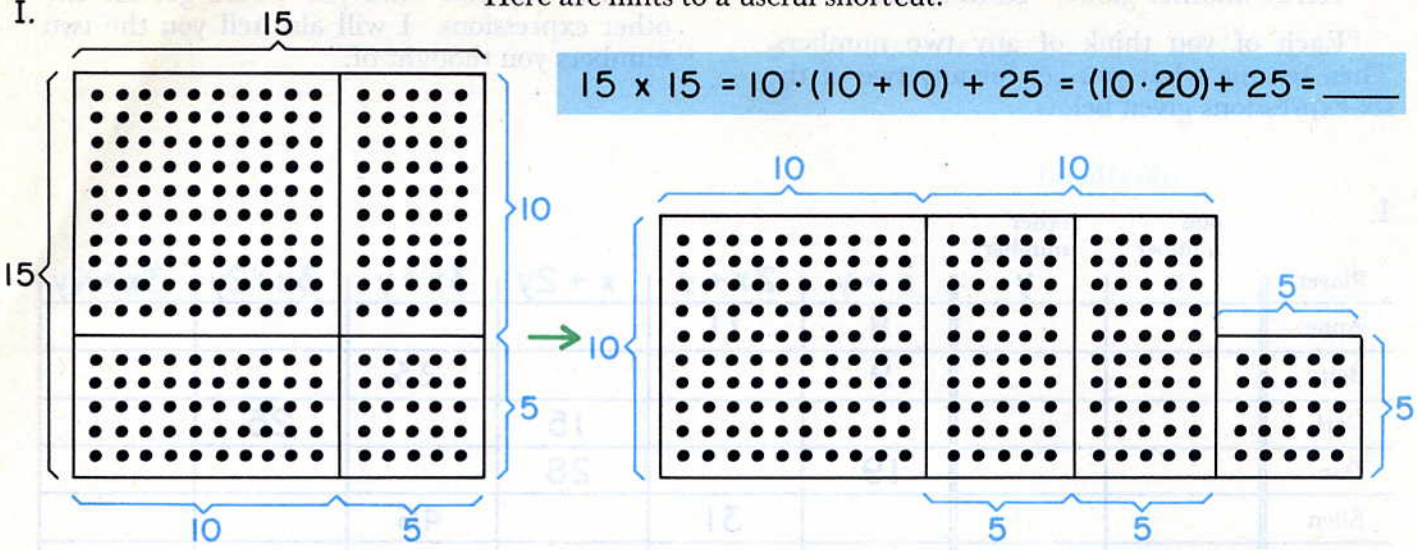
"Again think of any two numbers (one larger than the other). Tell me what you get as results for two of the five statements. I will tell you the numbers you thought of."

III.

Words	Shorthand	Anne	Betty	Carl	Dan	Ellen	Fred	Gil
Think of a number.	p							
Think of a smaller number.	q							
Add your numbers.		8	12		19	29		
Add the smaller number to twice the larger number.		14				49		
Subtract the smaller number from the larger number.			2	7	5		6	
From five times the larger number subtract the smaller number.				47			50	35
Add three times the larger number to twice the smaller number.								21

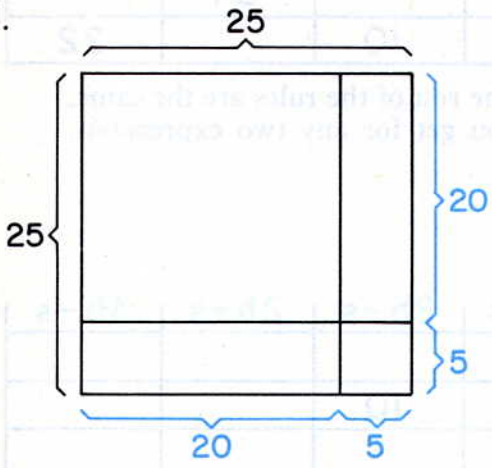
Here are hints to a useful shortcut.

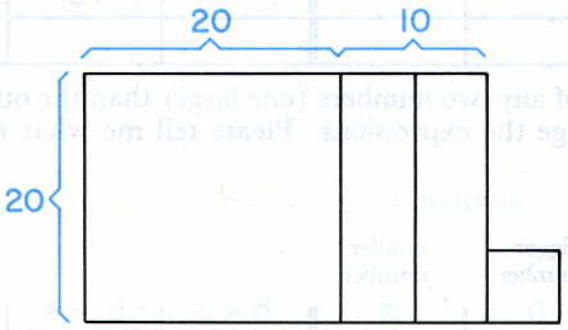
I.



$15 \times 15 = 10 \cdot (10 + 10) + 25 = (10 \cdot 20) + 25 = \_\_\_\_$

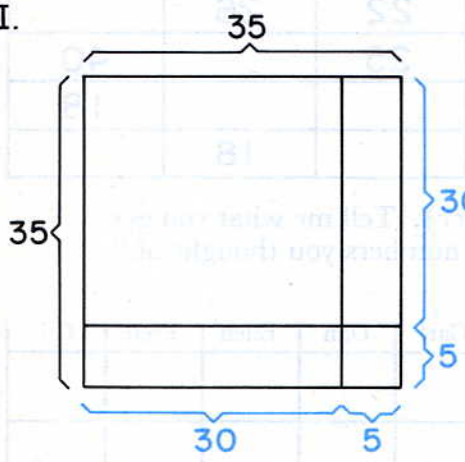
II.

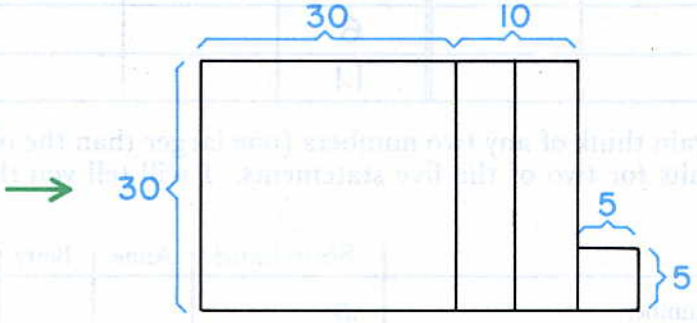




$25 \cdot 25 = 20 \cdot (20 + 10) + 25 = (20 \cdot 30) + 25 = \_\_\_\_$

III.





$35 \times 35 = 30 \cdot (30 + 10) + 25 = (30 \cdot 40) + 25 = \_\_\_\_$

IV.

a. $15 \cdot 15 = 10 \cdot (10 + 10) + 25 = (10 \cdot 20) + 25 =$ _____

b. $25 \cdot 25 = 20 \cdot (20 + 10) + 25 = (20 \cdot 30) + 25 =$ _____

c. $35 \cdot 35 = 30 \cdot (30 + 10) + 25 = (30 \cdot 40) + 25 =$ _____

d. $45 \cdot 45 = 40 \cdot (40 + 10) + 25 = (40 \cdot 50) + 25 =$ _____

e. $55 \cdot 55 = 50 \cdot (50 + 10) + 25 =$ _____ = _____

Can you guess the results for:

$65 \times 65 =$ _____

$85 \times 85 =$ _____

Test your guesses.

You can put the correct answers down as fast as you can write:

a.	b.	c.	d.	e.	f.	g.	h.	i.
15 x 15	25 x 25	35 x 35	45 x 45	55 x 55	65 x 65	75 x 75	85 x 85	95 x 95

Now put this useful shortcut to work by combining it with $(b+s)\cdot(b-s)=(b\cdot b)-(s\cdot s)$

1. $23\cdot 17=(20+3)\cdot(20-3)=(20\cdot 20)-(3\cdot 3)=400-9=$ _____
2. $27\cdot 23=(25+2)\cdot(25-2)=(25\cdot 25)-(2\cdot\ )=625-\ =$
3. $39\cdot 31=(35+4)\cdot(35-\ )=\ \cdot\ -\ \cdot\ =\ -\ =$
4. $41\cdot 39=\ =\ =\ =$
5. $26\cdot 24=\ =\ =\ =$
6. $47\cdot 43=\ =\ =\ =$
7. $98\cdot 92=\ =\ =\ =$
8. $109\cdot 101=\ =\ =\ =$
9. $82\cdot 78=\ =\ =\ =$
10. $61\cdot 49=(55+\ )\cdot(55-\ )=\ =\ =$

Now put this useful shortcut to work by combining it with $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$.

11. $25\cdot 26=25\cdot(25+1)=(25\cdot 25)+(25\cdot 1)=\ +\ =650$
12. $35\cdot 38=35\cdot(35+\ =\ \cdot\ +\ \cdot\ =\ +\ =$
13. $25\cdot 24=25\cdot(25-1)=(25\cdot 25)-(25\cdot 1)=\ -25=$
14. $45\cdot 43=45\cdot(\ -\ )=\ \cdot\ -\ \cdot\ =\ -\ =$
15. $39\cdot 35=35\cdot(\ +\ )=\ \cdot\ +\ \cdot\ =\ +\ =$

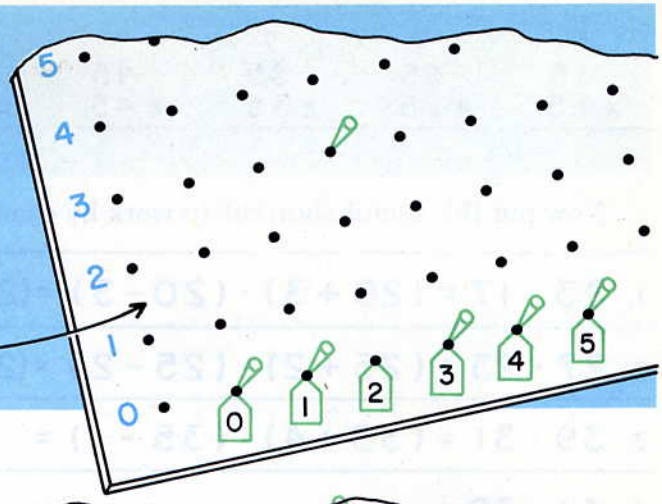
Now take most of the steps without writing them down — if you can.

A.	B.	C.	D.	E.	F.	G.
34 x 35	33 x 37	53 x 47	47 x 45	65 x 66	87 x 93	71 x 59
1225 − 35						
1,190						

I $35\cdot 35=$	III $35\cdot 36=$	V $34\cdot 36=$	VII $31\cdot 39=$
II $45\cdot 45=$	IV $46\cdot 45=$	VI $48\cdot 42=$	VIII $46\cdot 44=$

Using peg board, green pegs, and tags, Bill worked out a way of keeping track of the numbers people call out and the number he answers after applying his rule.

If someone calls out "2" and Bill answers "3," he finds the peg near the tag with 2 written on it. He moves the peg up to the row of holes opposite 3 on the left side of the board. It would look like the picture to the right.



"I'm thinking of a rule," Bill began.

"3," Ben called out.

"5," Bill answered. He recorded the answer. ⟶

"0," Helen called out.

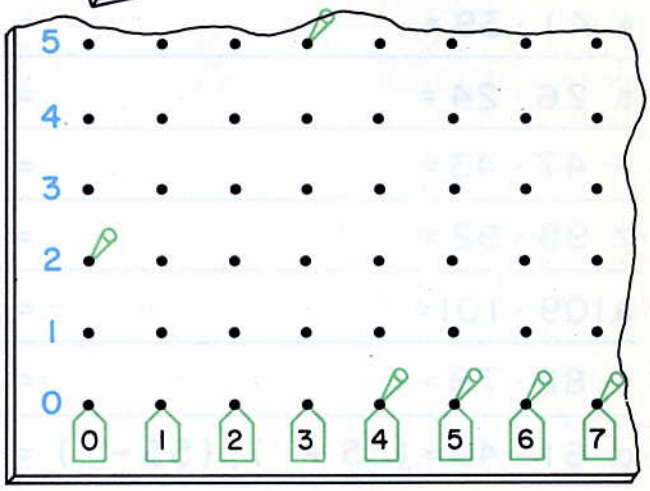
"2," Bill answered, and recorded the answer. ⟶

"1," Anne called out, "and I think you are going to answer 3." ⟶

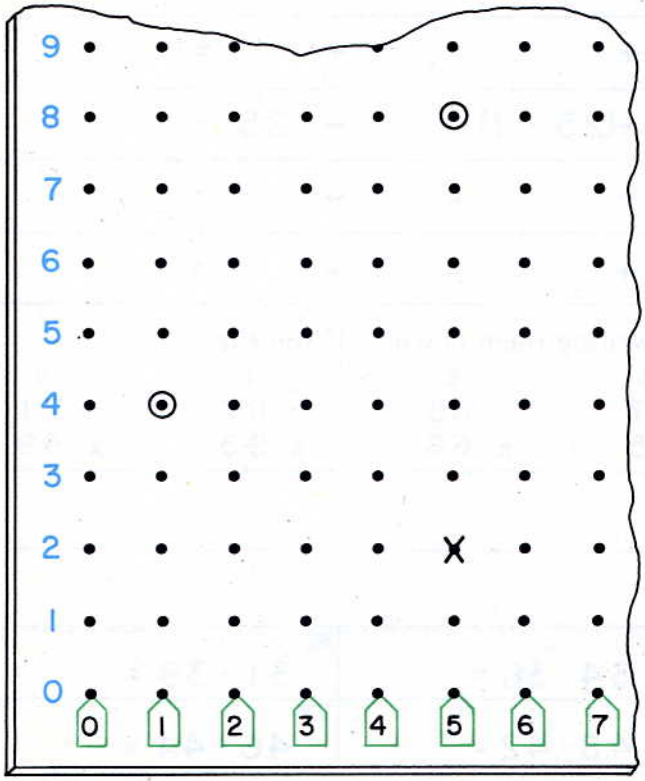
"That's right. You know my rule."

"If I say 2, will you say 4?" Al asked.

"Right. You know the rule also."



The first two results are recorded above. Please record the others.



On the left, keep a record of the games shown below. You will need to discover the secret rule for each game in order to complete it.

For the first game, instead of drawing pegs, circle the places where the pegs would go. The first two answers are already indicated for the first game. The first answer for the second game is indicated by an 'X'. Place 'X' marks where the pegs would be placed for the second game.

1.

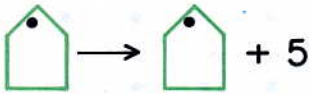
Numbers called out	1	5	4	0	6	3	2
Bill's answers	4	8	7				

2.

Numbers called out	5	3	6	0	4	1	2
Bill's answers	2	4	1				

Bill explains a shorthand way to write down rules after they are discovered.

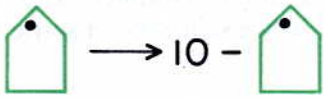
"When someone calls out a number, I imagine it as written on a tag at the bottom of the scoreboard. If my rule is to answer by saying a number that is five more than the number called out, I think this way:



"So, the green tag holds a place for any number you think of. I read that shorthand as:

'Tag goes to tag plus 5.'

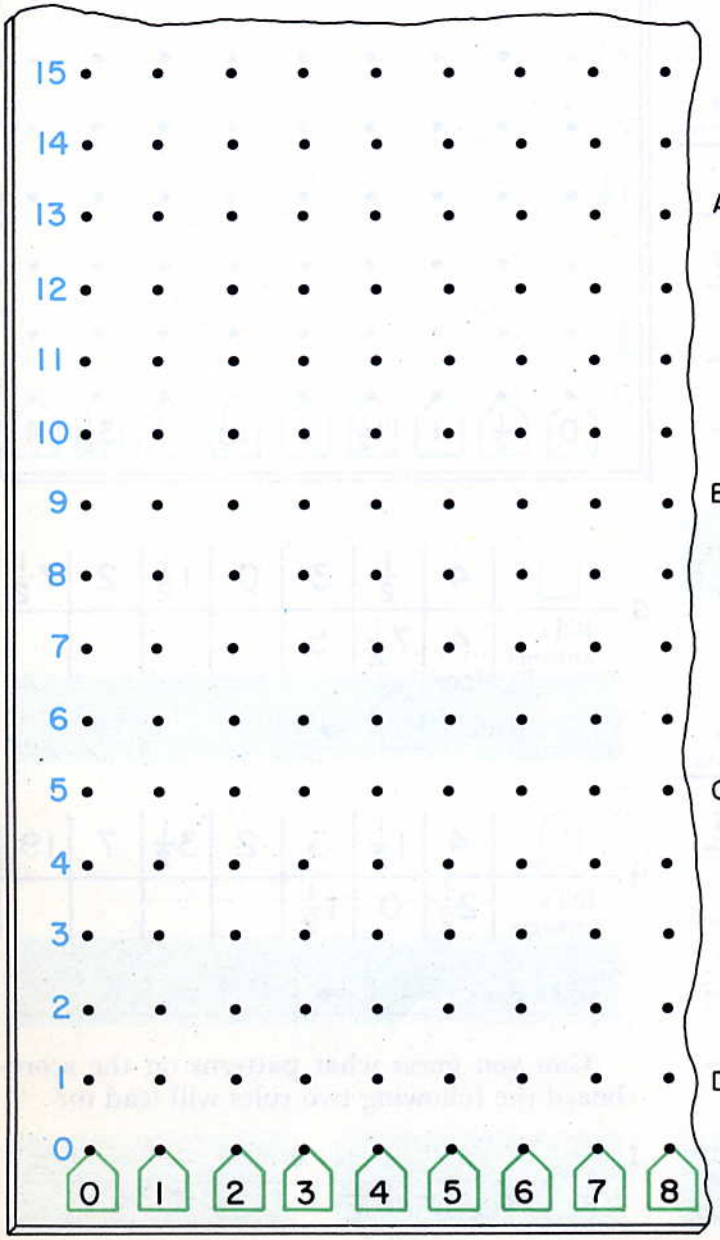
"Here's another rule. How much less than 10 is each number called out? In shorthand and words:



'Tag goes to 10 minus tag.'

Here are some games to finish. Record the results on the scoreboard and write the rule in Bill's shorthand. Some of the numbers called out and some of Bill's answers may be so large you can't record them on the small section of the scoreboard below. In that case, you will have to imagine where the pegs would go.

(Use the game letter to show where the pegs would go for each different game.)



A.

Numbers called out	5	8	2	1	4	7	20
Bill's answers	3	6	0	too small			

Bill's Rule:

B.

Numbers called out	3	8	0	6	5	19	7
Bill's answers	9	14					

Bill's Rule:

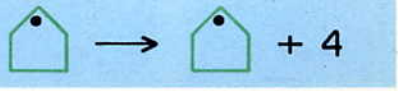
C.

Numbers called out	7	5	8	4	1	6	83
Bill's answers	3	1			too small		

Bill's Rule:

D.

Numbers called out	2	4	8	0	3	5	1
Bill's answers							

Bill's Rule: 

Bill changes the labels on his scoreboard.

"Thus far, we have been limited to whole numbers. I have changed the tags on the scoreboard so we can record some fractions . . .

"As we record the results on the scoreboard, watch for patterns. Sometimes the patterns will be very much like those on the previous page. Some will be different.

"I've started some games. Please discover the secret rule for each game."

A.

	Bill's answers
1	5
4	2
3	3
6	
$1\frac{1}{2}$	

Bill's Rule: ⌂ → _ − ⌂

B.

	Bill's answers
4	0
$2\frac{1}{2}$	$1\frac{1}{2}$
1	3
2	
$3\frac{1}{2}$	

Bill's Rule: ⌂ → _ − ⌂

C.

	Bill's answers
$\frac{1}{2}$	1
3	$3\frac{1}{2}$
0	$\frac{1}{2}$
4	
$15\frac{1}{2}$	

Bill's Rule: ⌂ → _ + ⌂

D.

	Bill's answers
$\frac{1}{2}$	$1\frac{1}{2}$
$1\frac{1}{2}$	$\frac{1}{2}$
2	0
0	
1	

Bill's Rule: ⌂ → _ − ⌂

E.

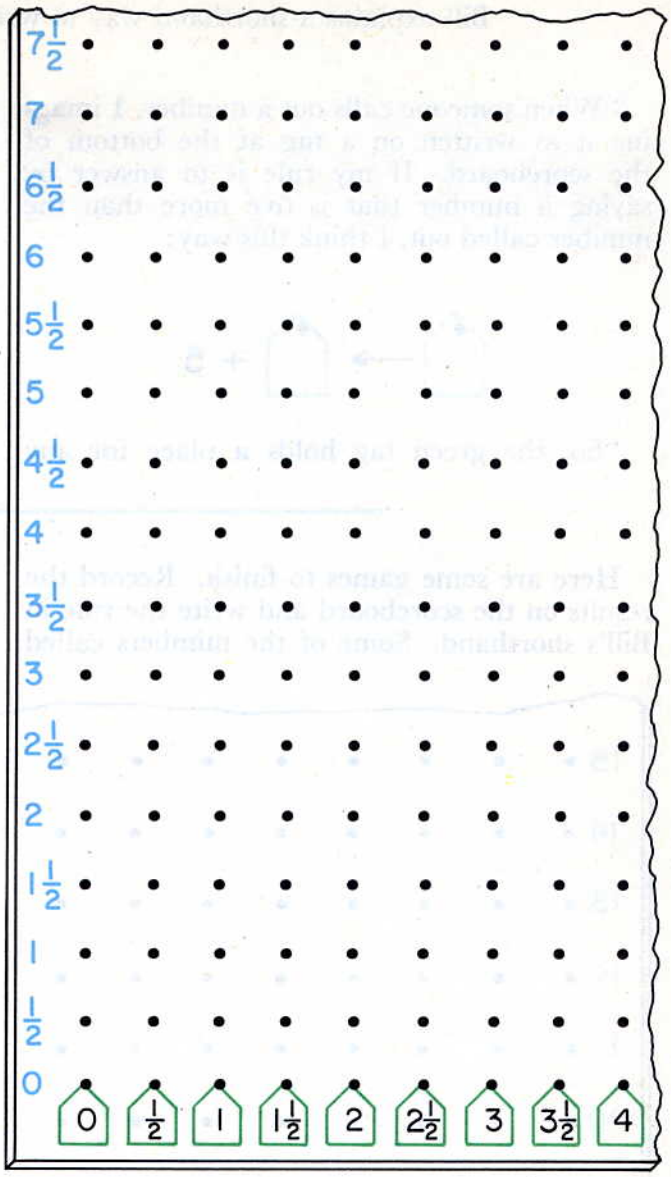
	Bill's answers
2	1
4	3
$1\frac{1}{2}$	$\frac{1}{2}$
3	
$2\frac{1}{2}$	

Bill's Rule: ⌂ → ⌂ − _

F.

	Bill's answers
0	$2\frac{1}{2}$
$\frac{1}{2}$	2
1	$1\frac{1}{2}$
$1\frac{1}{2}$	
2	

Bill's Rule: ⌂ → _ − ⌂



G.

⌂	4	$\frac{1}{2}$	3	0	$1\frac{1}{2}$	2	$7\frac{1}{2}$
Bill's answers	4	$7\frac{1}{2}$	5				

Bill's Rule: ⌂ →

H.

⌂	4	$1\frac{1}{2}$	3	2	$3\frac{1}{2}$	7	19
Bill's answers	$2\frac{1}{2}$	0	$1\frac{1}{2}$				

Bill's Rule: ⌂ →

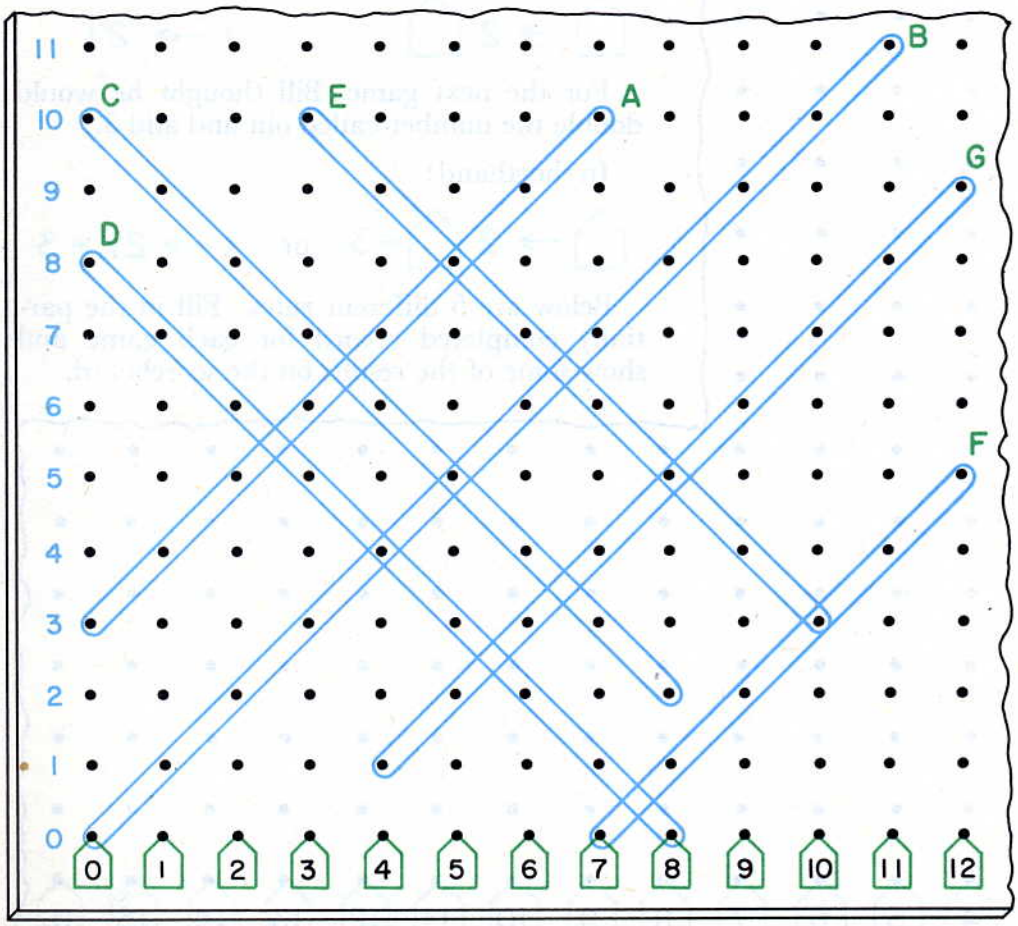
Can you guess what patterns on the scoreboard the following two rules will lead to?

I. ⌂ → ⌂ − $2\frac{1}{2}$

J. ⌂ → $3\frac{1}{2}$ − ⌂

Another way to play WHAT'S MY RULE?

"On this scoreboard," Bill explained, "I have arranged rows of pegs to indicate results of games played. I've put a rubber band around the results of each game. Each game is lettered. I've given the rule for Game A and listed a few results. Please find all the rules used and list 7 sample results for each game."



Rule A:

→	+3
	+3
7	10
5	8
0	3

Rule B:

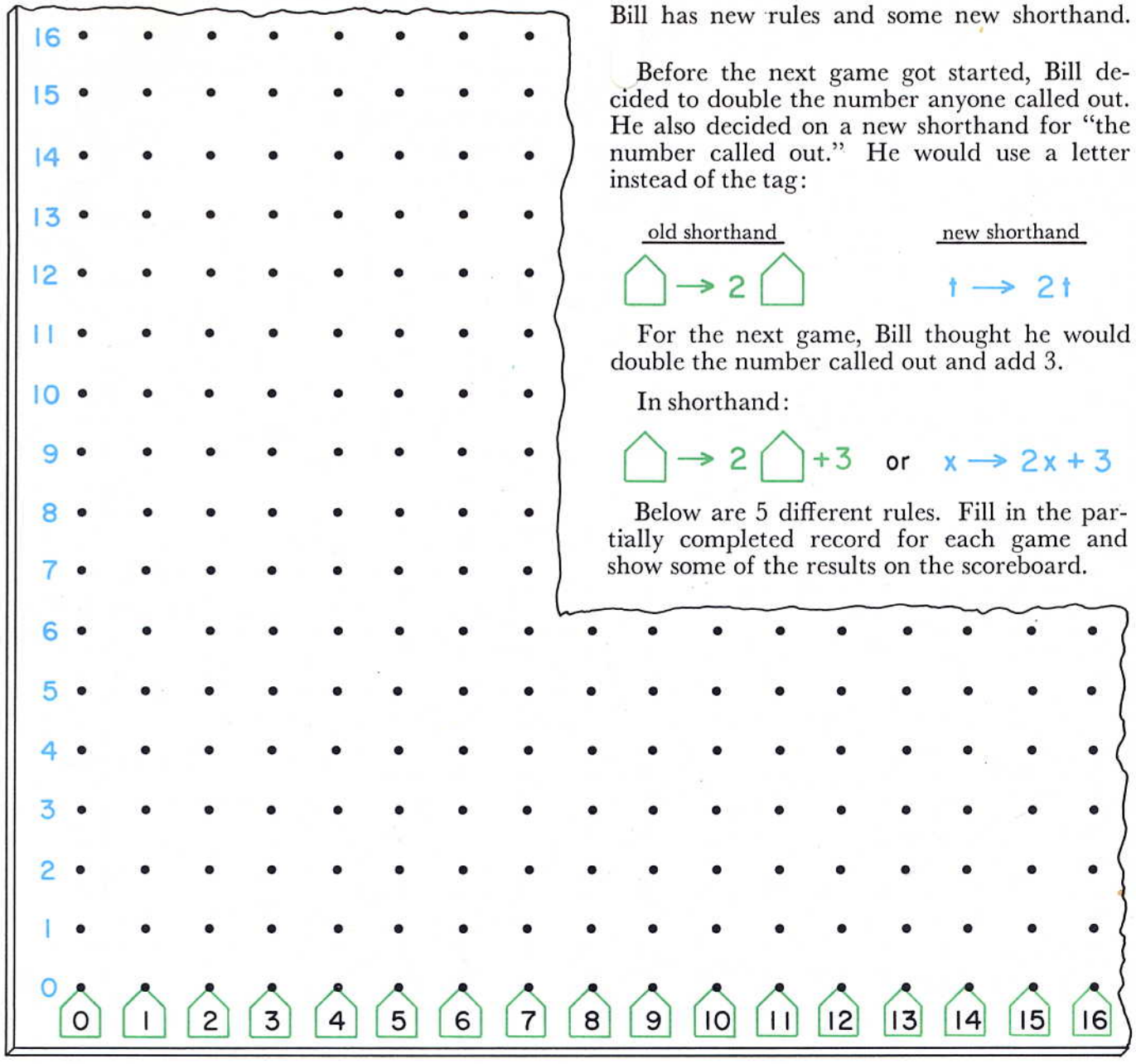
Rule C:

Rule D:

Rule E:

Rule F:

Rule G:



Bill has new rules and some new shorthand.

Before the next game got started, Bill decided to double the number anyone called out. He also decided on a new shorthand for "the number called out." He would use a letter instead of the tag:

old shorthand	new shorthand
→ 2	t → 2t

For the next game, Bill thought he would double the number called out and add 3.

In shorthand:

→ 2 +3 or x → 2x + 3

Below are 5 different rules. Fill in the partially completed record for each game and show some of the results on the scoreboard.

Bill's Rule J

t → 2t

4	8

Bill's Rule K

x → 2x + 3

1	5

Bill's Rule L

s → 4s

2	8

Bill's Rule M

r → r ÷ 2

6	3

Bill's Rule N

q → (q ÷ 2) - 3

8	1

Make up your own rules. Record game results on your own scoreboard.

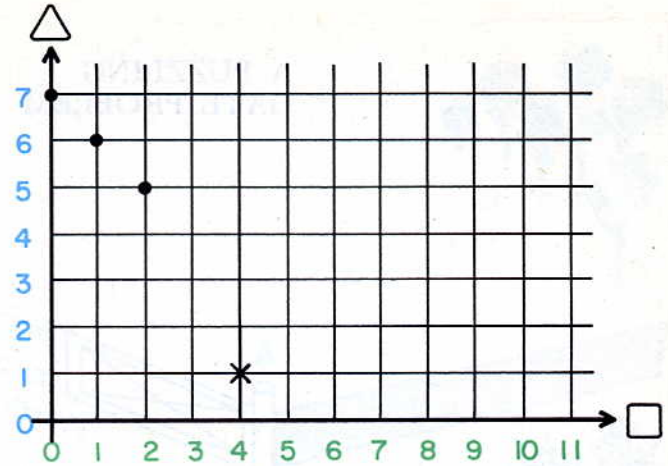
Fill in the partial record for each game and show some of the results on the scoreboard to the right of each game.

A. △ = 7 − □

□ →	0	1	2					
△ ↑	7	6	5					

B. △ = 5 − □

□ →	5	4	0	1	3	2		
△ ↑		1						



C. ◇ = ⬡

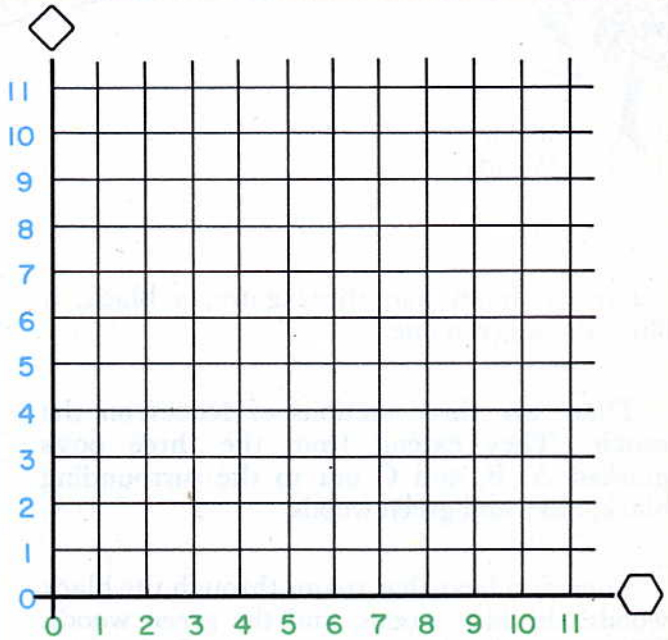
⬡ →	5	9	0	11				
◇ ↑					8	1		

D. ◇ = ⬡ − 3

⬡	3	4						
◇			2					

E. ◇ = ⬡ + 2

⬡	0	1	2	3				
◇								



F. $y = 2 \cdot n$

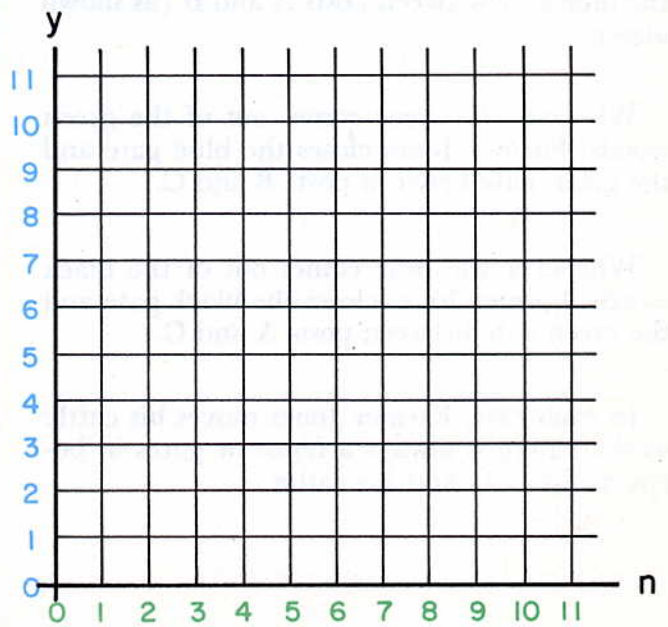
n →	0	1	2					
y ↑				6				

G. $y = 1 \cdot n$

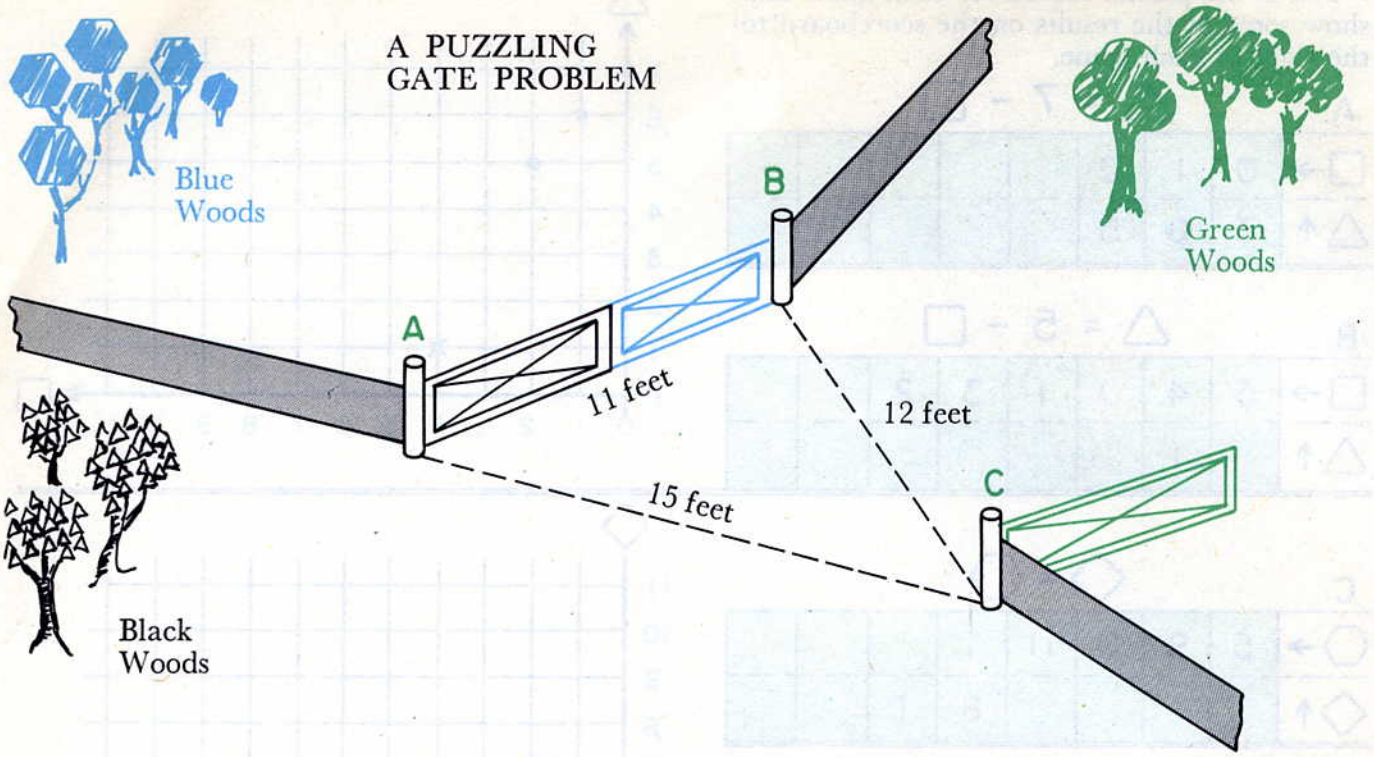
n								
y								

H. $y = \frac{1}{2} \cdot n$

n								
y	0	1	2	3	4			



A PUZZLING GATE PROBLEM



Farmer Jones had three gates, a black, a blue, and a green one.

There are three sections of fences on the ranch. They extend from the three posts marked A, B, and C out to the surrounding black, blue, and green woods.

There is a bear that roams through the black woods, the blue woods, and the green woods. Whenever the bear comes out of the blue woods, Farmer Jones closes the black gate and the blue gate between posts A and B (as shown above).

Whenever the bear comes out of the green woods, Farmer Jones closes the blue gate and the green gate between posts B and C.

Whenever the bear comes out of the black woods, Farmer Jones closes the black gate and the green gate between posts A and C.

In each case, Farmer Jones moves his cattle so that there is always a fence or gates in between the bear and his cattle.

The distance from post A to post C is 15 feet. The distance from post A to post B is 11 feet. The distance from post B to post C is 12 feet.

Each time the gates are closed, they meet without any gap between them, and they do not overlap.

How long are the gates?

The black gate is _____ feet long.

The blue gate is _____ feet long.

The green gate is _____ feet long.

Take your time. You may find this problem very difficult. Think about it. Try to solve it. If you don't discover the answers, don't think about the problem for a while.

Come back to the problem another day. Try to solve it again.

NOTE: If you solve the problem, please don't tell others. Each person should discover the answer for himself!

1 2 3 4 5 J 70 69 68 67 66